EECS 16B Designing Information Devices and Systems II
Spring 2019 Discussion Worksheet

5A: Identification

1. System identification by means of least squares

Working through this question will help you understand better how we can use experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares techniques you learned in 16A. You will later do this in lab for your robot car.

As you were told in 16A, least-squares and its variants are not just the basic workhorses of machine learning in practice, they are play a conceptually central place in our understanding of machine learning well beyond least-squares.

Throughout this question, you should consider measurements to have been taken from one long trace through time.

(a) Consider the scalar discrete-time system

$$x(i+1) = ax(i) + bu(i) + w(i) \tag{1}$$

Where the scalar state at time i is $x(i)$, the input applied at time i is $u(i)$ and $w(i)$ represents some external disturbance that also participated at time i.

Assume that you have measurements for the states $x(i)$ from $i = 0$ to m and also measurements for the controls $u(i)$ from $i = 0$ to $m-1$. **Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a and b.**

(b) What if there were now two distinct scalar inputs to a scalar system

$$x(i+1) = ax(i) + b_{1,1}u_1(i) + b_{1,2}u_2(i) + w(i) \tag{2}$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters $a, b_{1,1}, b_{1,2}$.

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

(d) **Returning to the scalar case with only one input, what could go wrong? When would you be unable to use least-squares to get the parameters you want?**

(e) Now consider the two dimensional state case with a single input.

$$\vec{x}(i+1) = \begin{bmatrix} x_1(i+1) \\ x_2(i+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}(i) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_1(i) + \vec{w}(i) \tag{3}$$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$? What work/computation can we reuse across the two problems?

(f) Based on everything that you have seen so far, **how would you use least-squares to estimate the parameters for a general $n \times n$ matrix A with k inputs whose impact is governed by a $k \times n$ matrix B? What work/computation can we reuse? What can go wrong?**

(g) In the parts so far, we had direct access to (possibly noisy) measurements of the true state $\vec{x}$. Let us consider if we instead had access to something else.

Consider the scalar discrete-time system in (1). Suppose that we had access to (possibly noisy) measurements of outputs $y(i)$ instead where

$$y(i) = cx(i). \tag{4}$$

Note that a, b, c are all unknown to us. **Can we, using measurements of $u(i)$ and outputs $y(i)$ figure out the three unknown parameters a, b, c without ambiguity? Why or why not? Is there anything we can figure out?**

(h) Consider now the same basic question for a two-dimensional system. Can we tell differently parameterized systems apart given only their input/output behavior?

Consider the following explicit examples:

System 1 (5)

$$\vec{x}(i+1) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{3} \end{bmatrix} \vec{x}(i) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(i) + \vec{w}(i) \tag{6}$$

$$\vec{y}(i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}(i) \tag{7}$$

System 2 (8)

$$\vec{\tilde{x}}(i+1) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \vec{\tilde{x}}(i) + \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix} u(i) + \vec{\tilde{w}}(i) \tag{9}$$

$$\vec{\tilde{y}}(i) = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \vec{\tilde{x}}(i) \tag{10}$$

System 3 (11)

$$\vec{\check{x}}(i+1) = \begin{bmatrix} \frac{5}{6} & 1 \\ -\frac{1}{6} & 0 \end{bmatrix} \vec{\check{x}}(i) + \begin{bmatrix} 1 \\ -\frac{1}{12} \end{bmatrix} u(i) + \vec{\check{w}}(i) \tag{12}$$

$$\vec{\check{y}}(i) = \begin{bmatrix} 1 & 0 \\ \frac{4}{3} & 4 \end{bmatrix} \vec{\check{x}}(i) \tag{13}$$

Please complete the following table to see how these different systems respond to the same input sequence

i	$u(i)$	$x_1(i)$	$x_2(i)$	$\tilde{x}_1(i)$	$\tilde{x}_2(i)$	$\check{x}_1(i)$	$\check{x}_2(i)$	$y_1(i)$	$y_2(i)$	$\tilde{y}_1(i)$	$\tilde{y}_2(i)$	$\check{y}_1(i)$	$\check{y}_2(i)$
0	+1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	0	1	1			1	$-\frac{1}{12}$						
2	0	$\frac{3}{4}$	$\frac{1}{3}$			$\frac{3}{4}$	$-\frac{1}{6}$						
3	2	$\frac{11}{24}$	$\frac{1}{9}$			$\frac{11}{24}$	$-\frac{1}{8}$						

Explain what is going on? When is it ambiguous?

(i) To avoid the ambiguity, we need to make some choices. When the dimension of $\vec{y}$ is the same as the underlying state $\vec{x}$ and we believe that the mapping is invertible, then we can just treat the $\vec{y}$ as though they were indeed the true states. However, our understanding of the concept of observability suggests that we might be able to proceed even if we didn't have full-dimensional measurements.

For example, consider (13) and suppose that we only had the first row of this — in other words, we only got to see the first state at each time. Suppose that we could assume the form for the transition matrix we see in (12) — namely that

$$A = \begin{bmatrix} a_{11} & 1 \\ a_{21} & 0 \end{bmatrix} \tag{14}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{15}$$

with $y(i) = x_1(i)$ at each time.

Unroll things appropriately and **set this up as a least-squares problem to learn the unknown parameters a_{11}, a_{21}, b_1, b_2.**

(j) For the case of a scalar input and a scalar output, **how would you hope to generalize the pattern you just found to the case of an n-dimensional state $\vec{x}$ that you cannot see?**

We will see later that this is indeed possible to do, as well as seeing how you could conceivably guess at what reasonable n might be in a data-driven way. This gets to the very heart of many ideas in modern machine learning.

2. Extra Practice: Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}[t+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[t]$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) Is the system controllable?

(b) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T-1$?

(c) Find the set of all possible states reachable after two timesteps.

(d) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T-1$?

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