

1 Stability

For some additional background on stability, see pages 9 through 15 of Prof. Murat Arcak's 16B Reader.

Suppose we have a discrete-time linear system with some disturbance, that is

$$\vec{x}(i+1) = A\vec{x}(i) + B\vec{u}(i) + \vec{w}(i) \quad (1)$$

in the general vector case. Until now, we have been essentially ignoring the disturbance $\vec{w}(i)$ out of convenience, that is treating the system (1) as if $\vec{w}(i) = 0$ for all i . However, we know that this is never true in practice! Whether it comes from true randomness in the system or through model imperfections, our system will always be affected by some disturbance.

What can disturbances do to our system? Well, the worst-case scenario is that the disturbance term $w(i)$ affects the system dynamics in such a way that the states $x(i)$ grow unboundedly with time. In most situations, having unbounded growth of the state is bad. Think of a car parked on top of a hill in neutral gear: a small push will cause its velocity to grow unboundedly, and the end result is decidedly not good.

This is where our notion of *stability* comes in. We will say that *a system is stable if $x(i)$ remains bounded for any initial condition and any bounded sequence of inputs $u(i)$* . This definition captures the intuitive notion that a small (i.e. bounded) disturbance should not be able to cause unbounded growth of the state.

1.1 Determining stability in discrete time systems

How can we tell if a system is stable or not? It would be nice if we could determine if a system is stable just from its representation (1). To start off with, we'll look at the scalar case, and see if we can figure out a rule. Consider the following three scalar systems:

- $x_1(i+1) = u_1(i) + w(i)$
- $x_2(i+1) = \frac{1}{2}x_2(i) + u_2(i) + w(i)$
- $x_3(i+1) = 2x_3(i) + u_3(i) + w(i)$

that are affected by some noise $w(i)$ that is *bounded*, that is $|w(i)| < \epsilon$ for all times i , for some ϵ .

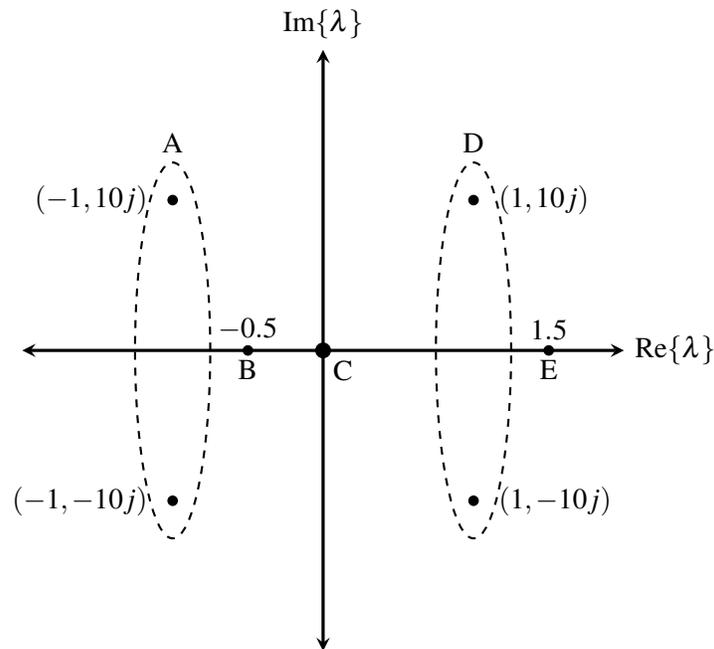
Suppose, furthermore, that all three have initial state $x_1(0) = x_2(0) = x_3(0) = x_0$, and based on this initial conditions we have designed input sequences $u_k(i)$ that bring the state to the origin in one time step. From $i = 1$ onwards, the state is only affected by the bounded disturbance. The question is, under these conditions, **which of the three states will remain bounded over time?**

The stability of these three systems, and your knowledge of sums of geometric series, should convince you that the following rule is valid: *for the scalar case $x(i+1) = \lambda x(i) + bu(i) + w(i)$, the system will be stable if $|\lambda| < 1$.*

For the vector case, a similar rule holds. Let λ be any particular eigenvalue of the matrix A in 1. Then *the vector system $\vec{x}(i+1) = A\vec{x}(i) + B\vec{u}(i) + \vec{w}(i)$ is stable is $|\lambda| < 1$ for all λ* . We will see why this is true later in the class.

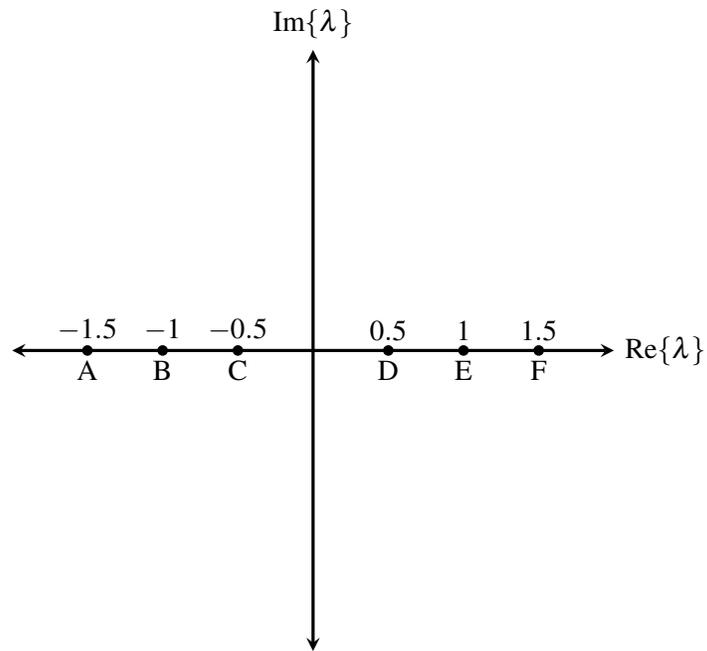
1. Continuous-time system responses

We have a system $\frac{d\vec{x}}{dt} = A\vec{x}$ with eigenvalues λ . For each set of λ values plotted on the real-imaginary axis, sketch $\vec{x}(t)$ with an initial condition of $x(0) = 1$. Determine if each system is stable.



2. Discrete time system responses

We have a system $x(k+1) = \lambda x(k)$. For each λ value plotted on the real-imaginary axis, sketch $x(k)$ with an initial condition of $x(0) = 1$. Determine if each system is stable.



3. Discrete-Time Stability

Determine which values of α and β will make the following discrete-time state space models stable:

(a)

$$x(t+1) = \alpha x(t)$$

(b)

$$\vec{x}(t+1) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \vec{x}(t)$$

(c)

$$\vec{x}(t+1) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \vec{x}(t)$$

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