1 Controller Canonical Form

When working with systems using a state-space representation, you may have noticed that a single system can be represented in many different forms, depending on lots of choices that we can make — such as how you ordered your state vector. Writing out systems in certain **canonical forms** often allows engineers to quickly determine system behavior.

The **controllable canonical form** for a sytem with a scalar input, which guarantees controllability and simplifies eigenvalue placement, takes the following form:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

You will verify later that this essentially means that every single system that is controllable can be viewed as having a higher-order recurrence-relation inside (for the discrete-time case) or a higher-order differential equation inside (for the continuous-time case).

The notation here is slightly different from lecture, but better matches the reader.

1.1 Change of Basis to Controller Canonical Form

Given a **controllable** system of the form $\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$, we can transform it into controllable canonical form by choosing some T, such that:

$$\vec{z} = T\vec{x}, \tilde{A} = TAT^{-1}, \text{ and } \tilde{B} = TB$$

for matrices $\tilde{A}$ and $\tilde{B}$ of the form shown above.

Here T^{-1} is the new basis that we are using. In that basis, the system looks like it is in controllable canonical form.

In lecture, you saw a way to compute this transformation as the composition of two different transformations. It is also possible to understand it more directly. Of course, the original controllability matrix still plays an essential role.

We can calculate this T using $R_n = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$, the controllability matrix of the original form using A and B. Note that R_n is invertible, because the original system is controllable, and we are restricting attention to the case of scalar inputs — so B is actually just a vector $\vec{b}$. Calculate R_n^{-1} — this maps vectors into the basis given by the columns of R_n. Therefore $\vec{a} = R_n^{-1}A^nB$ gives the coefficients of the characteristic polynomial of A.

With those coefficients in hand, we can compute a new matrix $\widetilde{R}_n = \begin{bmatrix} \widetilde{B} & \widetilde{A}\widetilde{B} & \cdots & \widetilde{A}^{n-1}\widetilde{B} \end{bmatrix}$, the controllability matrix of the CCF form using $\widetilde{A}$ and $\widetilde{B}$. It turns out that once again $\vec{a} = \widetilde{R}_n^{-1}\widetilde{A}^n\widetilde{B}$.

Then, we can cascade them together to get the transformation $T = \widetilde{R}_n R_n^{-1}$. This maps from the original coordinates to coordinates for where the CCF system representation lives.

Also notice that when we place our system in feedback using $u(t) = \tilde{K}\vec{z} = K\vec{x}$ with $\tilde{K} = KT^{-1}$, we get the closed loop matrix

$$T(A+BK)T^{-1} = \tilde{A} + \tilde{B}\tilde{K}$$

The eigenvalues of both systems are the same, and we can arbitrarily assign the eigenvalues of $\tilde{A} + \tilde{B}\tilde{K}$ with the choice of $\tilde{K}$. Therefore, we just proved that *controllability enables arbitrary eigenvalue assignment* in any state space system. Note that it is *not* necessary to bring the system to the controller canonical form to assign its eigenvalues. You can still use what we did in the last section to choose K in order to obtain desirable eigenvalues.

1.2 Coefficient Matching to Controllable Canonical Form

For smaller systems, we can use a technique called **coefficient matching** to bring the system to controllable canonical form. Different forms of a system are “similarity transforms” (i.e. the same linear operation, just represented in different coordinates) of each other. This means that the eigenvalues of the system are preserved. Thus, we can find the characteristic polynomial of our system in its original form, and then map the coefficients of this polynomial into the state matrix of our system in controller canonical form.

Let’s walk through an example to convert

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$

to controller canonical form

$$\vec{z}(t+1) = \tilde{A}\vec{z}(t) + \tilde{B}u(t) = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}\vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

Our first step should always be to **check for controllability** to ensure that a transformation into controllable canonical form actually exists.

$$R_n = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Now that we have verified controllability, we can calculate the characteristic polynomial of A.

$$\det(A - \lambda I) = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1$$

The characteristic polynomial of $\tilde{A}$ is

$$\det\left(\tilde{A} - \lambda I\right) = -\lambda(a_2 - \lambda) - a_1 = \lambda^2 - a_2\lambda - a_1$$

From here we can see that $a_2 = 2$ and $a_1 = -1$, giving us the controller canonical form

$$\vec{z}(t+1) = \tilde{A}\vec{z}(t) + \tilde{B}u(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

1. **Control Canonical Form - Introduction**

   (a) Show that for any square matrix A and any invertible matrix T, the matrix TAT^{-1} has the same characteristic polynomial as A.

   (b) (**VERY Optional**): Compute the characteristic polynomial of

   $$A = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{bmatrix}$$

   (c) Show that a discrete-time system in controllable canonical form is essentially a higher order scalar recurrence relation with scalar input.

   (d) Show that a continuous-time system in controllable canonical form is essentially a higher order scalar differential equation with scalar input.

   (e) Express the system

   $$\vec{x}(t+1) = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

   in controllable canonical form.

2. **Controllable Canonical Form - Eigenvalues Placement**

   Consider the following linear discrete time system

   $$\vec{x}(t+1) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

   (a) Is this system controllable?

   (b) Is the linear discrete time system stable?

   (c) Bring the system to the controllable canonical form

   $$\vec{z}(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

   using transformation $\vec{z}(t) = T\vec{x}(t)$

(d) Using state feedback $u(t) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \vec{z}(t)$ place the eigenvalues at $0, 1/2, -1/2$.

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