Discussion 7A

This discussion will focus on linear functions, its properties and how we can approximate non-linear functions, in a small neighborhoods, to behave as linear functions. So let's begin our discussion by understanding the properties of linear functions.

Linear Vs. Non-Linear Scalar Functions

Suppose we have $y = f(x)$, then we say that f is **linear** if and only if the following properties hold:

(a) it satisfies **Scaling**: for *any* number α, and *any* x, we have:

$$f(\alpha x) = \alpha f(x).$$

(b) AND: it satisfies **Superposition**: for *any* x and y, we have

$$f(x+y) = f(x) + f(y).$$

The above properties can be combined as follows

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Where the above statement is true for *any* α and β and *any* x and y.

Questions:

1. Check if the following functions are linear? Prove or disprove.

(a) $f(x) = 5x$.
(b) $g(x) = x^2$.

2. Suppose $f(x)$ is linear, show that $f(0) = 0$.

Quick Aside: If $f(x) = g(x) + k$, where $g(x)$ is linear and k is some constant (not equal to zero), then $f(x)$ is said to be **affine**.

Taylor Series

But most of the time, we will encounter non-linear functions in real life. So, how can we approximate these non-linear functions to follow the properties we introduced in the previous section? Let's begin by recalling the **Taylor series**.

Suppose we have a scalar function

$$y = f(x), \quad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

Next, let's pick an *expansion point* x^*. We can make the following statement:

If f is differentiable k times at x^* (for any $k \geq 1$), *i.e.*, $\frac{d^{(k)}}{dx^k}$ should be well defined.

Then, the Taylor series expansion [1] of $f(\cdot)$ to k terms is $f(x^*+\varepsilon) \approx f(x^*) + \varepsilon f'(x^*) + \varepsilon^2 \frac{f^{(2)}(x^*)}{2!} + \varepsilon^3 \frac{f^{(3)}(x^*)}{3!} + \cdots + \varepsilon^k \frac{f^{(k)}(x^*)}{k!}$.

The approximation is better when we are closer to x^*, *i.e.*, a small ε, and if k is large.

Questions:

1. Find the Taylor expansion of the following functions:

   (a) $f(x) = \sin(x)$.

   (b) $g(x) = e^x$.

Bonus: Choose $k = 5$, plot $\sin(x)$ and our above Taylor expansion. Also plot their error in a separate plot. What do you observe?

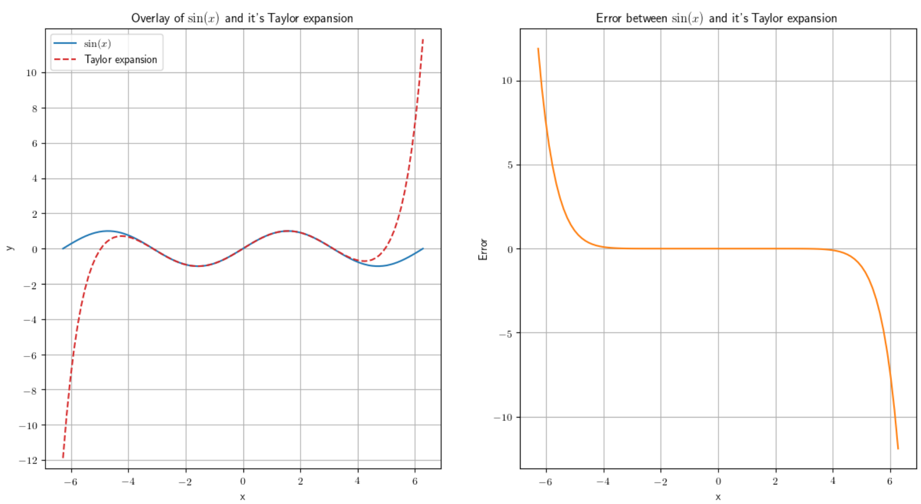

Figure 1: *Left*: Plot of $\sin(x)$ and its Taylor expansion upto $k = 5$ around the expansion point $x^* = 0$. *Right*: Error between the two functions.

Linearizing a Scalar Function at an Expansion Point

Now that we have familiarized ourselves with the Taylor expansion, how can we use this to linearize [2] functions? Well, if we set $k = 1$, then we have

$$f(x^*+\varepsilon) \approx f(x^*) + \varepsilon f'(x^*).$$

In the above expression, both $f(x^*)$ and $f'(x^*)$ are constant, and hence our approximation is only a function of ε. That's all there is to it!

[1] Here, $f^{(k)}$ defines the k^{th} derivative of $f(x)$.

[2] Here we use the term linearize a bit loosely, and we are tolerating our approximations being affine as well.

Questions:

1. Linearize the function $f(x) = \sin(x)$ around $x^* = 0, \frac{\pi}{2}$ and $\frac{\pi}{4}$.

   Bonus: Plot $\sin(x^* + \varepsilon)$ vs. the linear approximation for each of the above cases.

Linearizing Scalar Differential Equations

Now, let's put together the concepts we have learnt in the previous sections and understand how to linearize homogenous differential equations of the form

$$\frac{d}{dt}x = f(x).$$

Here we assume that $f(x)$ is non-linear to begin with, and that we are linearizing around the expansion point x^*.

Let's begin by looking at a neighborhood around x^*, *i.e.*, $x(t) = x^* + \delta_x(t)$. Plugging this back into our differential equation we get,

$$\begin{aligned} &\frac{d}{dt}(x^* + \delta_x(t)) = f(x^* + \delta_x(t)) \\ \Rightarrow &\cancel{\frac{d}{dt}x^*} + \frac{d}{dt}\delta_x(t) = f(x^* + \delta_x(t)) \\ \Rightarrow &\frac{d}{dt}\delta_x(t) = f(x^* + \delta_x(t)). \end{aligned}$$

Next, we need to make an important assumption to justify our linearization. Let $\delta_x(t)$ be small enough such that $f(x^* + \delta_x(t)) \approx f(x^*) + f'(x^*)\delta_x(t)$ is reasonably accurate. Using this linearization, we get

$$\frac{d}{dt}\delta_x(t) \approx f(x^*) + f'(x^*)\delta_x(t).$$

Finally, suppose we had chosen to linearize our differential equation around an equilibrium point [3] of the differential equation, then $f(x^*) = 0$. Hence, the linerization of our original differetial equation about the equilibrium point is

$$\frac{d}{dt}\delta_x(t) = f'(x^*)\delta_x(t).$$

An important point to note here is that our linear differential equation is no longer a function of 'x' but rather its distance from x^*.

Questions:

1. Linearize $\frac{d}{dt}x = -\sin(x)$ around $x^* = 0$ and solve the linearized differential equation. Next, verify that the linearization satisfies the 'smallness' assumption, *i.e.*, if $\delta_x(t)$ is small enough for $t \geq 0$.

2. Linearize the above differential equation around $x^* = \pi$. Does the linearization hold?

Contributors:

[3] Alternatively known as the DC operating point or fixed point.

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