This homework is due on Wednesday, February 20, 2019, at 11:59PM.
Self-grades are due on Saturday, February 23, 2019, at 11:59PM.

1. **Complex algebra**

   In this question, we will review our understanding of complex numbers.

   (a) **Express the current waveform**

   $$i(t) = -0.5\cos\left(3\pi\times 10^9 t + 30^\circ\right)\text{mA}$$

   in the standard cosine form and then determine its amplitude, frequency and phase angle and $i(t)$ at $t = 0.1$ ns. (The standard cosine form has the form of $\alpha\cos(\omega t+\theta)$, where α is positive and θ is an angle between 0 and 2π in radians or 0 and 360 in degrees.)

   (b) A 4 kHz sinusoidal voltage waveform $v(t)$, with a 8 V amplitude, was observed to descend through the value of 4 V at $t = 1$ ms. **Determine the standard cosine form of $v(t)$.**

   (c) **Express $z_1 = (-1-j)^{\frac{1}{2}}$ in polar form.** Is the answer unambiguous?

   (d) Given $z = 1.8 - j3.6$, **determine e^{z^*+1} in polar form.**

   (e) **Transform $i(t) = 12\cos(6t-45^\circ) - 4\sin(6t)$ into the single complex number that is its phasor counterpart.**

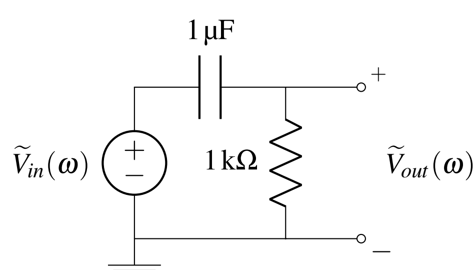
2. **Bode Plot Practice**

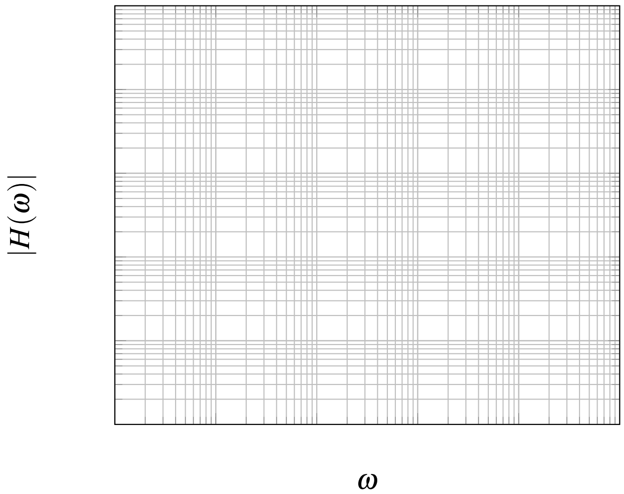
   The purpose of this exercise is to give you some extra practice with transfer functions and Bode plots. It is related to an exercise from one of your discussions.

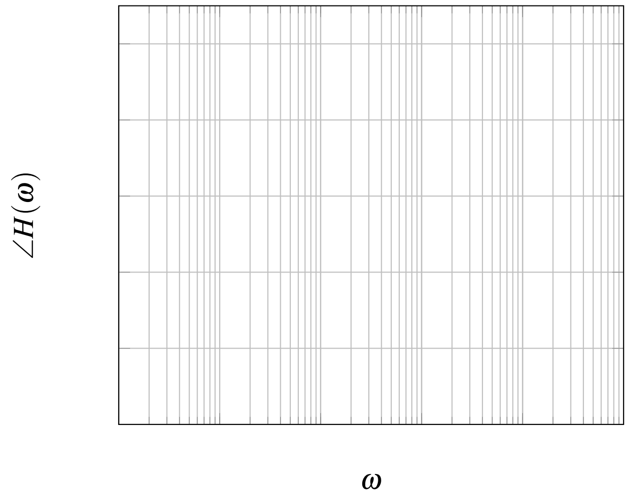
   For each of the three circuits shown below, please do the following:

   - Compute the transfer function $H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$ using the circuit analysis method of your choice;
   - Compute the magnitude and phase of the transfer function, that is $|H(\omega)|$ and $\angle H(\omega)$.
   - Draw a Bode plots approximations for $|H(\omega)|$ and $\angle H(\omega)$. At the end of this document are three blank log-log plots which you may use for this purpose.

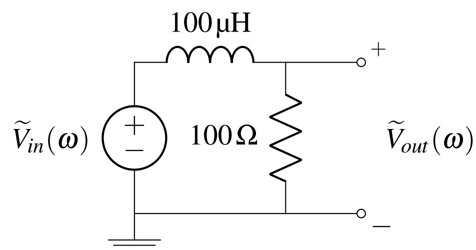
   (a) Given the following circuit, **complete the three steps above**.

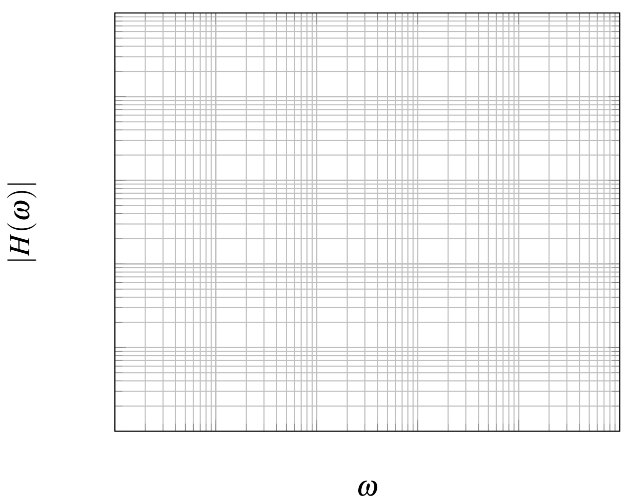


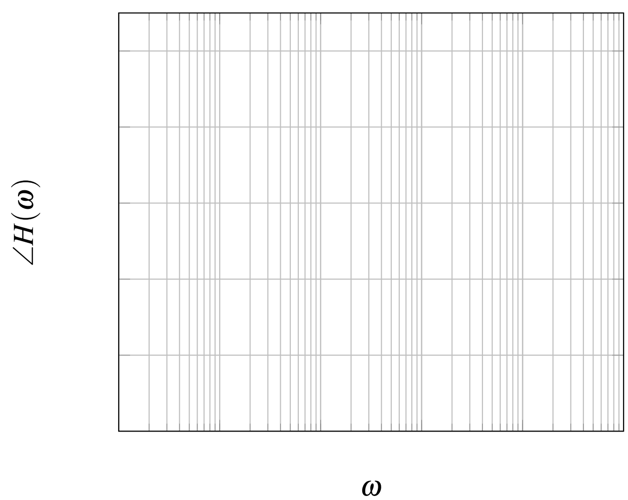




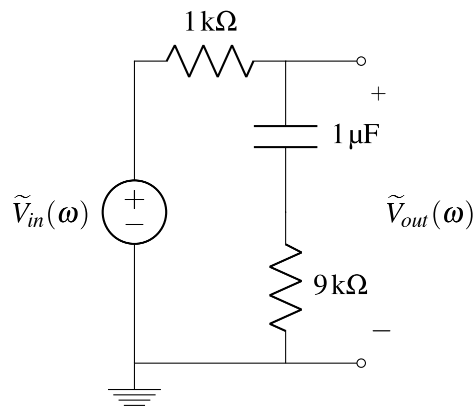
(b) Given the following circuit, **complete the three steps above**.

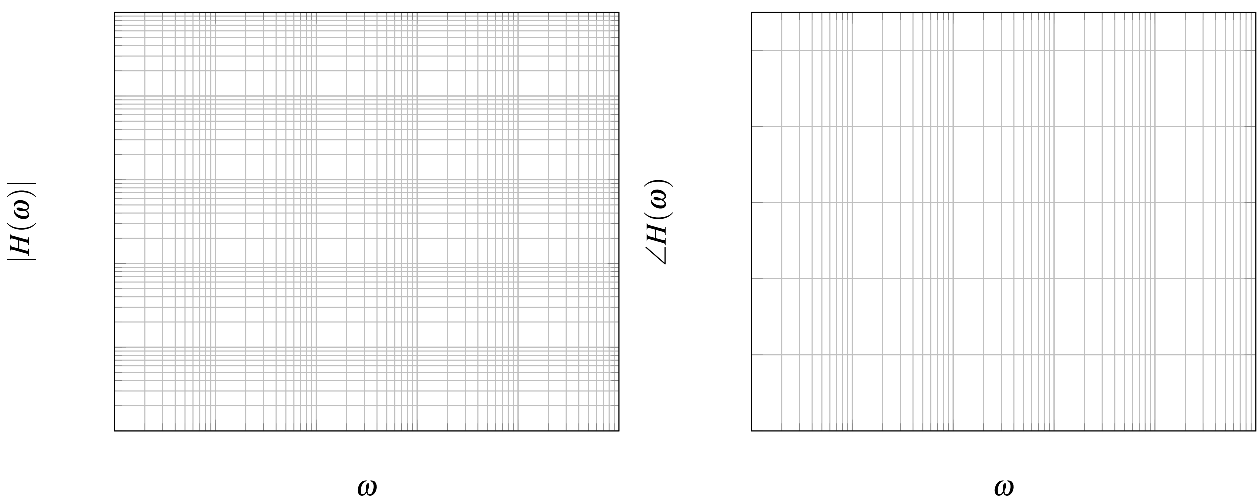






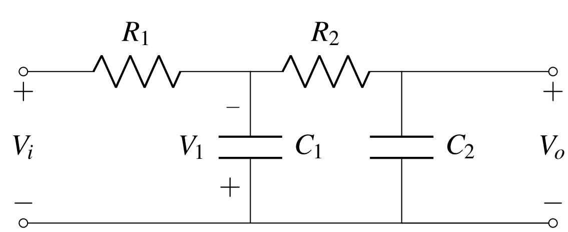
(c) Given the following circuit, **complete the three steps above**.





3. Transfer functions

Consider the circuit below.



The circuit has an input phasor voltage V_i at frequency ω rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage V_o at output terminals.

(a) We are going to construct the transfer function $H(\omega) = \frac{V_o}{V_i}$ in two steps. We will compute two intermediate transfer functions, $H_1(\omega) = \frac{V_1}{V_i}$ and $H_2(\omega) = \frac{V_o}{V_1}$. Then, we will find the overall transfer function as the product of these two intermediate transfer functions, i.e. $H(\omega) = \frac{V_o}{V_i} = H_1(\omega)H_2(\omega)$.

For the first step, **find the intermediate transfer function $H_2(\omega) = \frac{V_o}{V_1}$.** Have your expression be in terms of Z_{R2} and Z_{C2}, that is the impedances of R_2 and C_2.

(b) Now, **compute the other intermediate transfer function,** $H_1(\omega) = \frac{V_1}{V_i}$. Have your expression be in terms of Z_{R1}, Z_{R2}, Z_{C1}, and Z_{C2}. *hint: Applying KCL at the V_1 node would be a good place to start. You should try to find an expression for H_1 that has factors that H_2 can cancel out.*

Then, **use these two to calculate the overall transfer function** as $H(\omega) = \frac{V_o}{V_i} = H_1(\omega)H_2(\omega)$.

(c) **Obtain an expression for $H(\omega) = V_o/V_i$ in the form**

$$H(\omega) = \frac{V_o}{V_i} = \frac{1}{1 + 2\xi\frac{j\omega}{\omega_c} + \frac{(j\omega)^2}{\omega_c^2}},$$

given that $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, $C_1 = \frac{9}{2}\,\mathrm{F}$, and $C_2 = 1\,\mathrm{F}$. What are the values of ξ and ω_c?

(d) We can express the transfer function $H(\omega)$ in the polar form. That is,

$$H(\omega) = M(\omega)e^{j\phi(\omega)}$$

The functions $M(\omega)$ and $\phi(\omega)$ are the magnitude and the phase angle of $H(\omega)$, respectively. **Write down $M(\omega)$ and $\phi(\omega)$ using the transfer function you derived in part (b).**

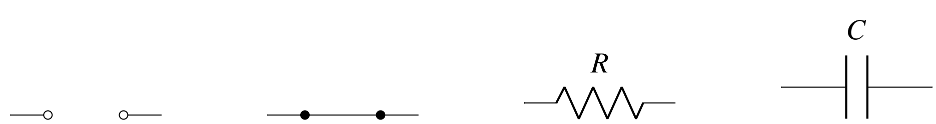
(e) **Draw Bode Plots of $|H(\omega)|$ and $\angle H(\omega)$.** A blank plot is provided here for you to use.

4. Circuit Design

In this problem, you will find a circuit where several components have been left *blank* for you to fill in.

Assume that the op-amp is *ideal*. *A special note on op amps in frequency domain analysis: The op-amps you learned about in 16A can be used in exactly the same way for setting up differential equations and even Phasor analysis in 16B. Treat them as ideal op-amps and invoke the Golden Rules.*

You have at your disposal *only one of each* of the following components (not including R_1 and R_2):



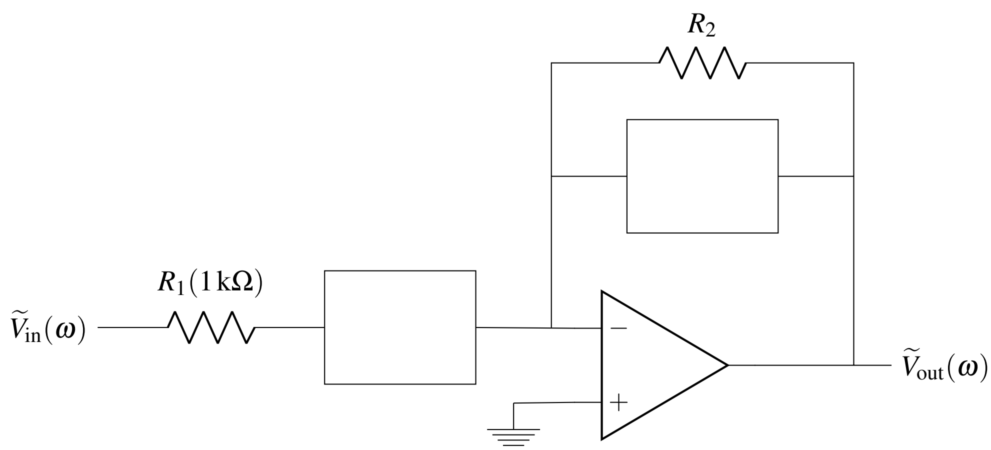
(a) an open circuit

(b) a short circuit

(c) a resistor (you choose from the values $R = 1\,\mathrm{k}\Omega, 15\,\mathrm{k}\Omega, 30\,\mathrm{k}\Omega$)

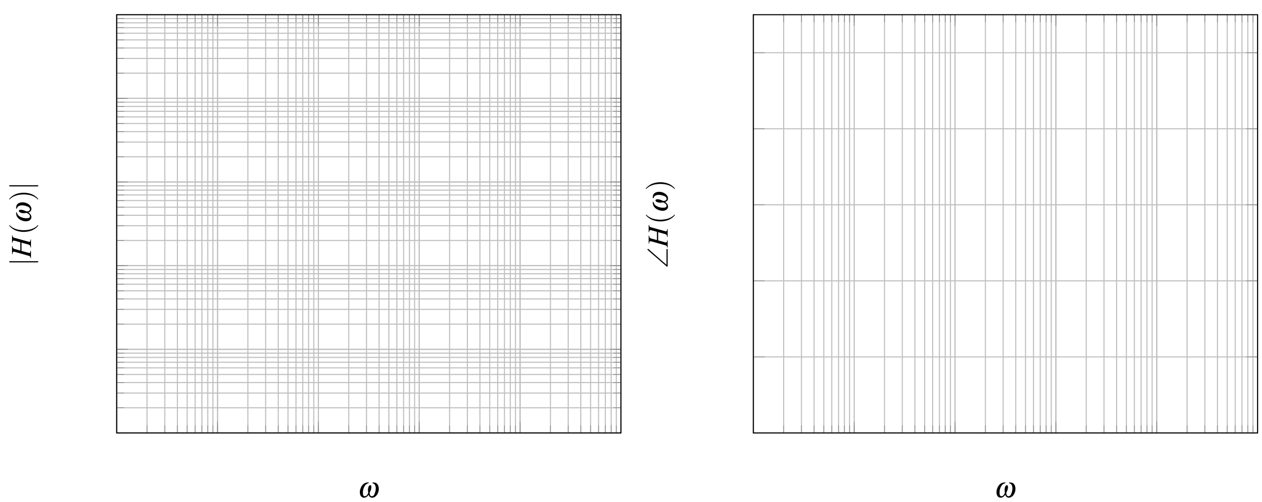
(d) a capacitor (you choose from the values $C = 0.5\,\mu\mathrm{F}, 1\,\mu\mathrm{F}, 2\,\mu\mathrm{F}$)

Consider the circuit below. The voltage source $\widetilde{V}_{\text{in}}(t)$ has the form $\widetilde{V}_{\text{in}}(t) = v_0 \cos(\omega t + \phi)$. The labeled voltages $\widetilde{V}_{\text{in}}(\omega)$ and $\widetilde{V}_{\text{out}}(\omega)$ are the phasor representations of $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$. The transfer function $H(\omega)$ is defined as $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$.



(a) Let R_1 be $1\,\text{k}\Omega$. **Fill in the boxes and determine the value of R_2, such that:**
- It is a high-pass filter.
- $|H(\infty)| = 10$.
- $|H(10^3)| = \sqrt{50}$.
- R_2 must be one of the three values listed above.

(b) **Draw the Bode plot of this transfer function.** A blank plot is provided here for you to use.



5. Bass-booster

RC circuits and filters are very useful for altering the frequency content of signals. For example, audio equalization equipment can use these filters to adjust the pitch content of audio signals. Suppose we want to boost the bass of our favorite music, we can use the active filter circuit below to tune the frequency content of our favorite jams.

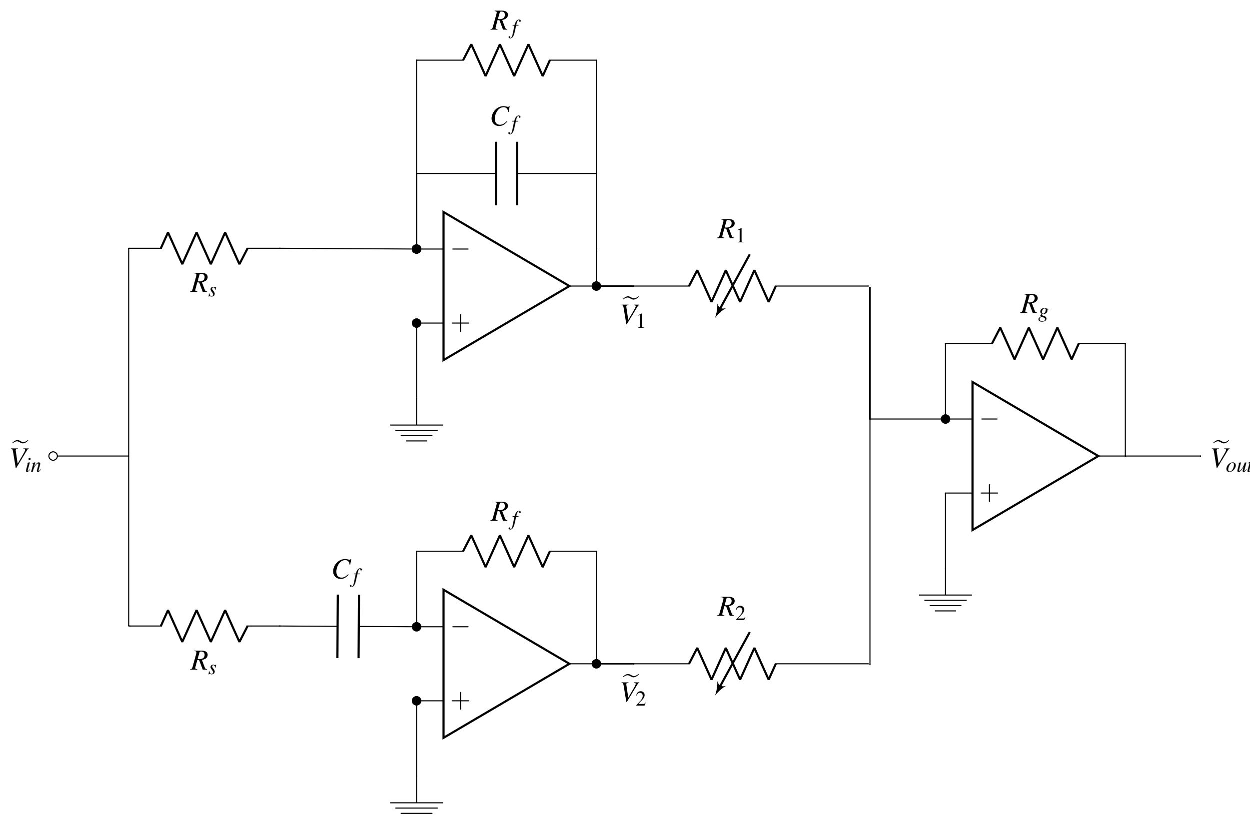


Figure 2: Audio Equalizer Circuit

$C_f = 400$ nF, $R_f = 1\text{k}\Omega$, $R_g = 100\Omega$, and $R_s = 1\text{k}\Omega$. R_1 and R_2 are both variable resistors that can be used to tune the frequency balance of our output signal. Assume the input audio signal is $v_{in}(t) = \cos(\omega t)$.

(a) First treat the first two active filters as disconnected and independent from the third, like in the figure below.
Find expressions for $H_1(\omega)$**=** $\frac{\widetilde{V}_1}{\widetilde{V}_{in}}$ **and** $H_2(\omega) = \frac{\widetilde{V}_2}{\widetilde{V}_{in}}$**,** where $\widetilde{V}_{in}$ and $\tilde{V}_1$, and $\tilde{V}_2$ are the phasor transformations of the voltages labeled on the circuit.
Sketch the bode plots for the magnitude of these transfer functions.
What kind of filter is each transfer function?

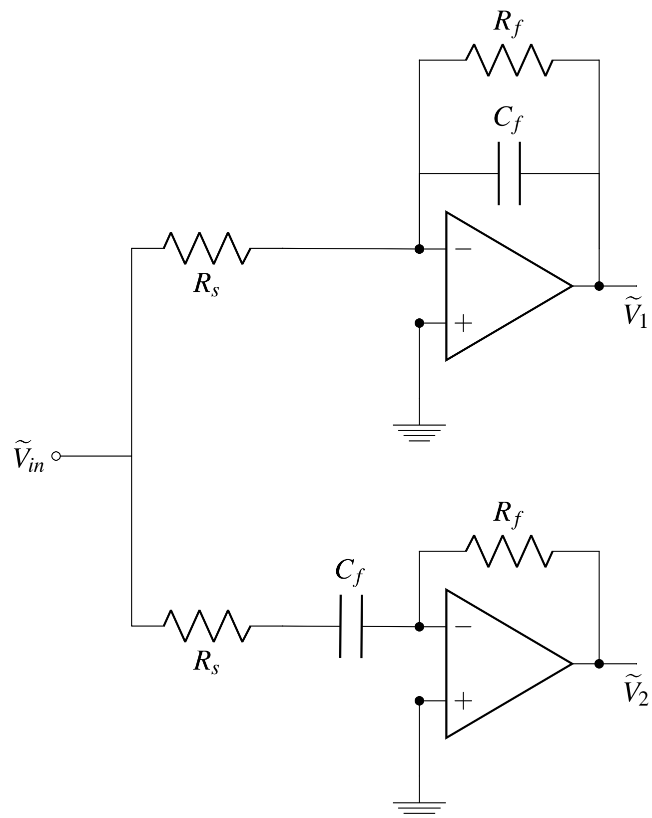


Figure 3: First Stage of Audio Equalizer

(b) **If we connect the outputs, $\widetilde{V}_1$ and $\widetilde{V}_2$, to R_1 and R_2, respectively, will $\widetilde{V}_1$ and $\widetilde{V}_2$ change? Can we use our expressions for $\widetilde{V}_1$ and $\widetilde{V}_2$ that we found in the previous part to represent $\widetilde{V}_1$ and $\widetilde{V}_2$ in the full three op amp circuit?**

(c) **Find a function that describes $\widetilde{V}_{out}$ in terms of $\widetilde{V}_1$ and $\widetilde{V}_2$.**

(d) **Combine the results of the last two parts to find an overall transfer function for $H_{ov}(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$.**

(e) **Using this circuit, determine values for R_1 and R_2 to boost our bass frequency signals ($f < 400$ Hz) without affecting mid and treble range signals?**

(f) **Sketch a Bode plot of the magnitude and phase of your overall bass-boosting function.** A blank plot is included here for you to use.

6. Designing a filter with phase constraints

You have a sensor which produces signals from 0.1 rad/s to 1 krad/s. For our purposes, the sensor looks just like a voltage source. In addition to the signal, the sensor produces an interfering voltage at 1 Mrad/s. We can model the sensor-plus-disturbance as:

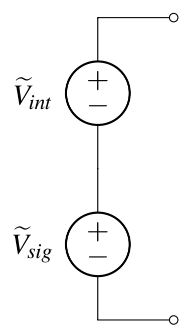


Figure 4: Model of the sensor, with interfering signal.

In this figure, $\widetilde{V}_{sig}$ is the voltage phasor of our signal, and $\widetilde{V}_{int}$ is the voltage phasor of the interference. Although we don't explicitly state the frequencies in phasor-land, remember that *these two phasors correspond to sinusoids of different frequencies*. This isn't a problem, just something to keep in mind.

Now, for the business at hand. We can remove the interference from the desired signal by adding a filter to the output of the sensor. For this problem, **your task is to design such a filter**. In fact, you will be designing *two* filters, and comparing their performance. However, you must be careful that the filter you design does not distort the signals we want to keep!

Your goal is to **design a filter which meets the following criteria**:

- the filter does not introduce more than 6 degrees of phase shift in any sensor input signal, over the entire range; 0.1 rad/s $< \omega <$ 1 k rad/s.
- the filter attenuates the noise source at 1 M rad/s as much as possible.

Your first filter design will be a single-pole RC filter. The circuit setup will be the following:

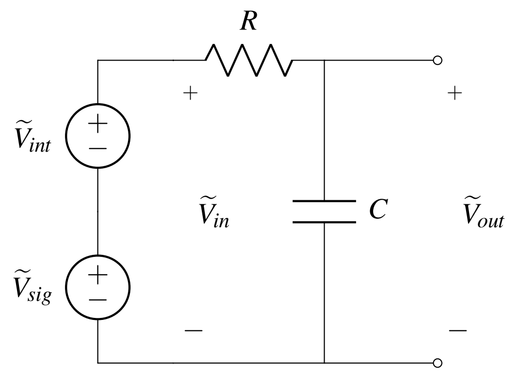


Figure 5: The sensor, augmented with your first filter design. We're counting on you!

The following questions will guide you through the process of designing the first filter.

(a) Write the transfer function of this filter in the form

$$H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{1}{1 + j\frac{\omega}{\omega_p}}, \tag{1}$$

where the quantity ω_p is called the *pole frequency*.

to maximize the attenuation of the noise, should you try to minimize the pole frequency or maximize the pole frequency?

(b) what is the minimum pole frequency that you can choose in order to meet the phase shift criterion?

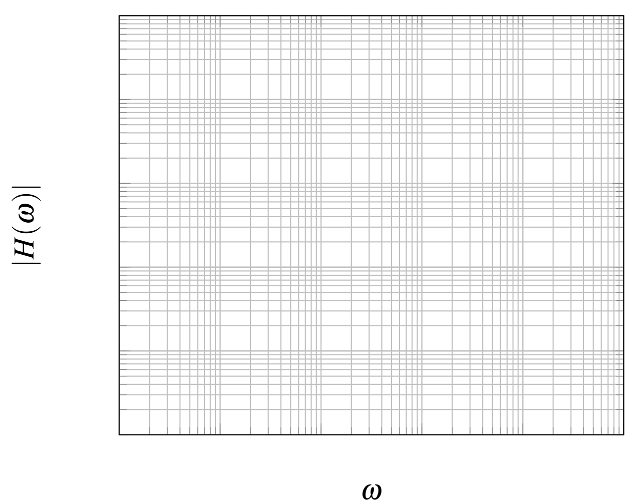
(c) The senior technician tells you that **the filter resistor needs to be** $100\,\Omega$. The reason for this particular value is to minimize *thermal noise*, the details of which are beyond the scope of this course. **Choose the capacitor to get the right pole frequency**.

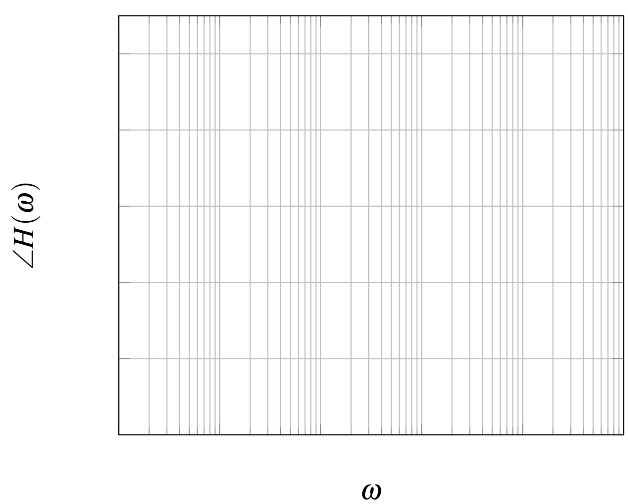
(d) What is the phase shift of a 1 krad/s signal, accurate to 1° or 10 mrad?

What is the magnitude of the attenuation of a 1 krad/s signal, accurate to 2 significant figures?

What is the magnitude attenuation of a 1 Mrad/s signal, accurate to 2 significant figures?

(e) **draw the straight-line approximation Bode plot of your filter**. Add in dots for the calculations that you did in part (d). A blank plot is provided here for you to use.





Comment on where there is a lot of error, and where there is not much.

(f) Now design a two pole filter by composing two RC low pass filters. The circuit setup will be the following:

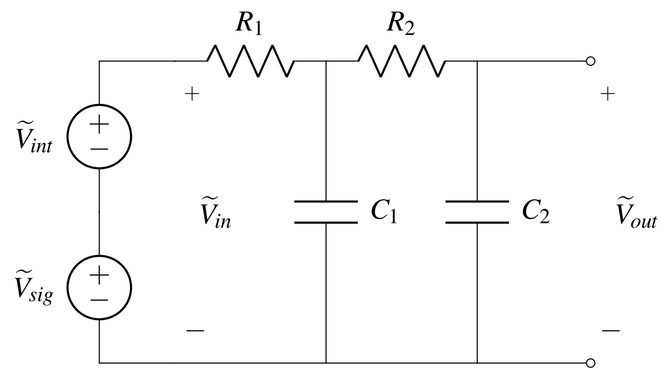


Figure 6: Model of the sensor, augmented with your second filter.

Make both poles of each section be at the same frequency. In other words, **your design must satisfy** $R_1C_1 = R_2C_2$.

what is the minimum pole frequency that you can choose in order to meet the phase spec? Be careful– it will have changed from part (a).

Note: for the remainder of this problem, **assume that the transfer function for this filter can be written as**

$$H(\omega) = \frac{1}{\left(1 + j\frac{\omega}{\omega_p}\right)^2}, \tag{2}$$

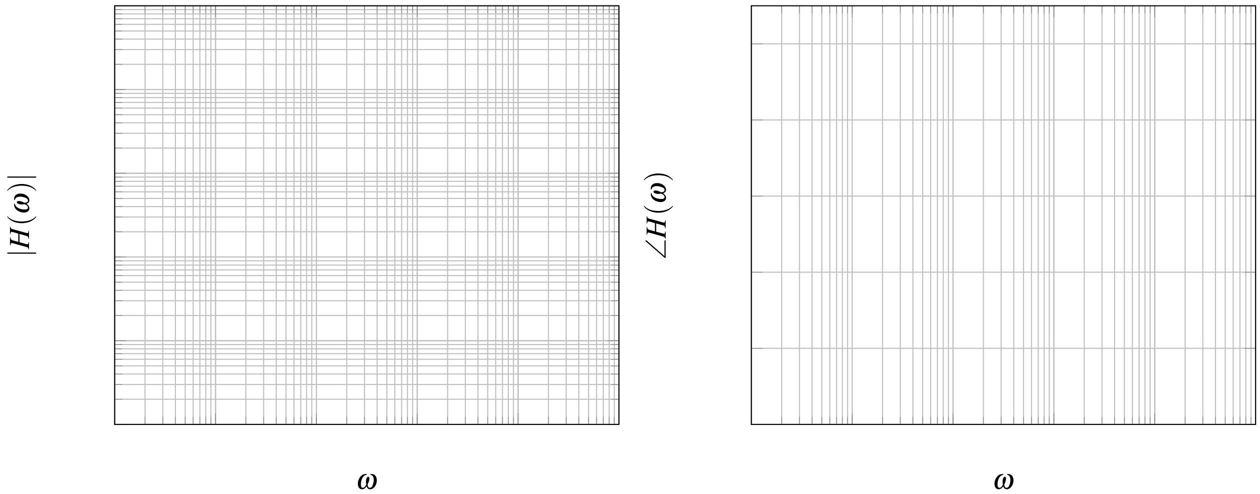
where ω_p is the pole frequency that we will choose. *This is only an approximation of the true transfer function!* However, in a later part of this problem, we will choose circuit components that make sure it is a good approximation.

(g) you pick the filter resistor in the first stage to be 100 Ohms– if you don't, the senior technician will get cranky again. **Choose the first stage capacitor to get the right pole frequency**.

(h) choose the resistor and capacitor for the 2nd stage filter so that the impedance of the second stage is always at least 100 times greater than the impedance of the first stage capacitor– that is, so that Z_{R2} is always 100 times greater than Z_{R1}, and Z_{C2} is always 100 times greater than Z_{C1}.

Using these component values, calculate the true transfer function $H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$. Use this expression to briefly explain why the approximation we made in part (f) is an accurate one.

(i) draw the straight-line approximation Bode plot of your filter. A blank plot is provided for you to use.



From the Bode plot,what is the attenuation of the 1 Mrad/s interference signal?

7. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

8. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) What sources (if any) did you use as you worked through the homework?

(b) Who did you work on this homework with? List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) How did you work on this homework? (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

(d) Roughly how many total hours did you work on this homework?

Contributors:

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