EECS 16B Designing Information Devices and Systems II
Spring 2019 A. Sahai, J. Roychowdhury, K. Pister HW 4

This homework is due on Wednesday, February 27, 2019, at 11:59PM.
Self-grades are due on Saturday, March 2, 2019, at 11:59PM.

1. Simple scalar differential equations driven by an input

In class, you learned that the solution to the simple scalar first-order differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{1}$$

with initial condition

$$x(t=0) = x_0 \tag{2}$$

is given by

$$x(t) = x_0 e^{\lambda t}. \tag{3}$$

In an earlier homework, you proved that these solutions are *unique*– that is, that $x(t)$ of the form in (3) are the only possible solutions to the equation (1) with the specified initial condition (2).

In this question, we will extend our understanding to differential equations with inputs.

In particular, the scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) \tag{4}$$

where $u(t)$ is a known function of time from $t = 0$ onwards.

(a) Suppose that you are given an $x_g(t)$ that satisfies both (2) and (4) for $t \geq 0$.
Show that if $y(t)$ also satisfies (2) and (4) for $t \geq 0$, then it must be that $y(t) = x_g(t)$ for all $t \geq 0$.
(HINT: You already used ratios in an earlier HW to prove that two things were necessarily equal. You might want to go for some variety in how you approach this. But be sure to leverage what you already proved earlier instead of having to redo all that work.)

(b) Suppose that the given $u(t)$ starts at $t = 0$ (it is zero before that) and is a nicely integrable function (feel free to assume bounded and continuously differentiably with bounded derivative — whatever conditions you assumed in your calculus course when considering integration and the fundamental theorem of calculus). Let

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \tag{5}$$

for $t \geq 0$.
Show that the $x_c(t)$ defined in (5) indeed satisfies (4) and (2).
Note: the τ here in (5) is just a dummy variable of integration. We could have used any letter for that local variable. We just used τ because it visually reminds us of t while also looking different. If you think they look too similar in your handwriting, feel free to change the dummy variable of integration to another symbol of your choice.
(HINT: Remember the fundamental theorem of calculus that you proved in your calculus class and manipulate the expression in (5) to get it into a form where you can apply it along with other basic calculus rules.)

(c) **Use the previous part to get an explicit expression for $x_c(t)$ for $t \geq 0$ when $u(t) = e^{st}$ for some constant s, when $s \neq \lambda$ and $t \geq 0$.**

(d) **Similarly, what is $x_c(t)$ for $t \geq 0$ when $u(t) = e^{\lambda t}$ for $t \geq 0$.**
(HINT: Don't worry if this seems too easy.)

2. Color Organ Filter Design

In the fourth lab, we will design low-pass, band-pass, and high-pass filters for a color organ. There are red, green, and blue LEDs. Each color will correspond to a specified frequency range of the input audio signal. The intensity of the light emitted will correspond to the amplitude of the audio signal.

(a) First, you remember that you saw in lecture that you can build simple filters using a resistor and a capacitor. **Design the first-order *passive* low- and high-pass filters with following frequency ranges for each filter using $1\,\mu\text{F}$ capacitors.** ("Passive" means that the filter does not require any power supply to operate on the input signal. Passive components include resistors, capacitors, inductors, diodes, etc., while an example of an active component would be an op amp).

- Low-pass filter: - cut-off frequency $f_c = 2400\,\text{Hz}$, $\omega_c = 2\pi \cdot 2400 \frac{rad}{sec}$
- High-pass filter: - cut-off frequency $f_c = 100\,\text{Hz}$, $\omega_c = 2\pi \cdot 100 \frac{rad}{sec}$

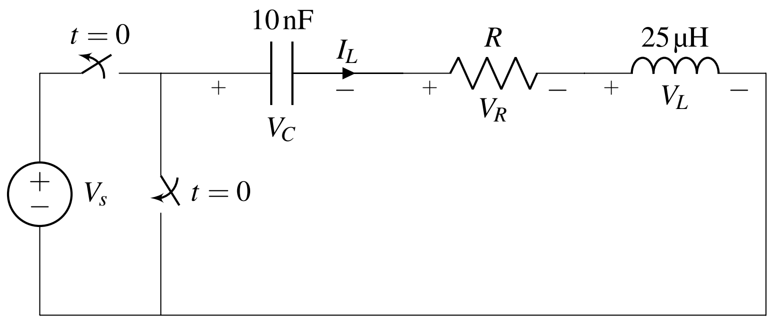
Show your work to find the resistor values that create these low- and high-pass filters. Draw the schematic-level representation of your designs. Please mark V_{in}, V_{out}, and the ground node(s) in your schematic. Round your results to two significant figures.

(b) You can build a bandpass filter by cascading the first-order low-pass and high-pass filters you designed in part (a). To do this, connect the V_{out} node of your low-pass filter directly to the V_{in} node of your high-pass filter. The V_{in} of your new band-pass filter is the V_{in} of your original low-pass filter, and the V_{out} of the new filter is the V_{out} of your original high-pass filter. **What is H_{BPF}, the transfer function of your new band-pass filter? Use R_L, C_L, R_H, and C_H to label the low-pass filter and high-pass filter components, respectively. Show your work.**

3. General RLC Responses

Note: This problem seems longer than it is. There are many short parts just to guide you.

Consider the following circuit:



Assume the circuit above has reached steady state for $t<0$. At time $t=0$, the switch changes state and disconnects the voltage source, replacing it with a short.

(a) **Write the system of differential equations in terms of state variables $x_1(t)=I_L(t)$ and $x_2=V_C(t)$ that describes this circuit for $t\geq 0$. Do not plug in the representative values for the capacitors and inductors just yet, leave the system symbolic in terms of V_s, L, R, and C.**

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$.**

This should be in the form $\frac{d}{dt}\vec{x}(t)=A\vec{x}(t)$.

(c) **Find the eigenvalues of the A matrix symbolically.**

(d) **Under what condition on the circuit parameters are there going to be a pair of distinct real eigenvalues of A?**

(e) **Under what condition on the circuit parameters are there going to be a pair of purely imaginary eigenvalues of A?**

(f) **Assuming that the circuit parameters are such so that there are a pair of (potentially complex) eigenvalues λ_1,λ_2 so that $\lambda_1\neq\lambda_2$, find eigenvectors $\vec{v}_{\lambda_1},\vec{v}_{\lambda_2}$ corresponding to them.**

(HINT: Rather than trying to find the relevant nullspaces, etc., you might just want to try to find eigenvectors of the form $\begin{bmatrix}1\\?\end{bmatrix}$ where we just want to find the missing entry. Can you see from the structure of the A matrix why we might want to try that guess?)

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V=[\vec{v}_{\lambda_1},\vec{v}_{\lambda_2}]$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t)=V\vec{\widetilde{x}}(t)$. **What is the $\widetilde{A}$ so that $\frac{d}{dt}\vec{\widetilde{x}}(t)=\widetilde{A}\vec{\widetilde{x}}(t)$?** It is fine to have your answer expressed symbolically using λ_1,λ_2.

(h) Suppose $R=1\,\text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s=1$ Volt. **Find the initial conditions for $\vec{\widetilde{x}}(0)$.**

(i) Continuing the previous part, **find $x_1(t)$ and $x_2(t)$ for $t\geq 0$.**

Note: Because there is a lot of resistance, this is called the "overdamped" case. However, at this particular point in this problem, you probably have no intuition for what is "over" about it.

(j) Suppose $R=0\,\text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s=1$ Volt. **Find the initial conditions for $\vec{\widetilde{x}}(0)$.**

(k) Continuing the previous part, **find $x_1(t)$ and $x_2(t)$ for $t \geq 0$.**

(l) Continuing the previous part, **are the waveforms for $x_1(t)$ and $x_2(t)$ "transient" — do they die out with time?**

Note: Because there is no resistance, this is called the "undamped" case.

(m) Now suppose that $R = 1\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ Volt. **Find the initial conditions for $\vec{\widetilde{x}}(0)$.**

(n) Continuing the previous part, **find $x_1(t)$ and $x_2(t)$ for $t \geq 0$.**
(HINT: Remember that $e^{a+jb} = e^a e^{jb}$.)

(o) Continuing the previous part, **are the waveforms for $x_1(t)$ and $x_2(t)$ "transient" — do they die out with time?**

Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

(p) **For the cases above where you got answers in terms of complex exponentials, why did the final voltage and current waveforms end up being purely real?**

(q) **Under what condition on the circuit parameters is there going to be a single eigenvalue of A?**

(r) **When there is a single eigenvalue of this particular matrix A, what is the dimensionality of the corresponding eigenspace?** (i.e. how many linearly independent eigenvectors can you find associated with this eigenvalue?) For this part, assume the given values for the capacitor and the inductor. It is easier to do the algebra with a non-symbolic matrix to work with.

(s) For a new coordinate system V, pick the first $\vec{v}_\lambda$ as being the eigenvector you get following the rule in an earlier part. For the second vector, just pick $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This implicitly defines variables $\vec{\widetilde{x}}$ in the transformed coordinates so that $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **What is the resulting $\widetilde{A}$ matrix defining the system of differential equations in the transformed coordinates?**

(t) Notice that the second differential equation for $\frac{d}{dt}\widetilde{x}_2(t)$ in the above coordinate system only depends on $\widetilde{x}_2(t)$ itself. There is no cross-term dependence. **Compute the initial condition for $\widetilde{x}_2(0)$ and write out the solution to this scalar differential equation for $\widetilde{x}_2(t)$ for $t \geq 0$.**

(u) With an explicit solution to $\widetilde{x}_2(t)$ in hand, substitute this in and **write out the resulting scalar differential equation for $\widetilde{x}_1(t)$.** This should effectively have an input in it.

Note: this is just the differential-equations counterpart to the back-substitution step that you remember from learning Gaussian Elimination in 16A, once you had done one full downward pass of Gaussian Elimination. You went upwards and just substituted in the solution that you found to remove this dependence from the equations above. This is the exact same pattern.

(v) **Solve the above scalar differential equation with input and write out what $\widetilde{x}_1(t)$ is for $t \geq 0$.**
(HINT: You might want to look at an earlier problem in this homework for help with this.)

(w) **Find $x_1(t)$ and $x_2(t)$ for $t \geq 0$ based on the answers to the previous three parts.**
This particular case is called the "critically damped case" for the RLC circuit.

(x) A sum of decaying exponentials is eventually dominated by the one that decays the slowest (just like how a sum of growing exponentials is eventually dominated by the one that grows the fastest). The imaginary part of the exponent in an exponential doesn't make something decay faster or slower — only the real part controls the rate of growth and decay.

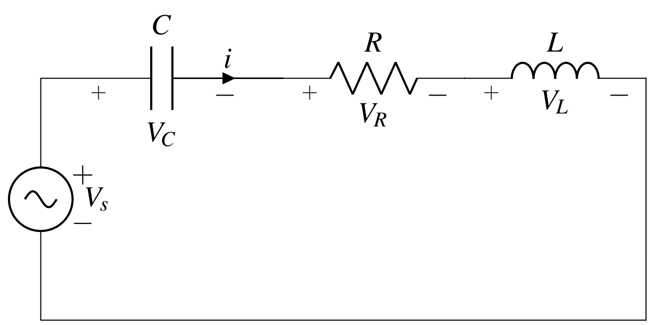
When the eigenvalues of the A matrix in the above RLC circuit are imaginary or complex, how does the real part of the eigenvalues behave as you decrease R?

(y) To see the impact of changing the parameters R and C, **play with the included Jupyter notebook**. See what happens above and below the critically damped condition. **Comment on what you observed.**

Note: The curve evaluation code in the included notebook has been slightly obfuscated from the approach taken in the parts above for solving the differential equations. So, it is not really going to be that useful to read those details of the code. You are, of course, free to try out your own expressions by editing the code as well as to add plots for individual parts — like separating out the contributions that are coming from each of the underlying modes for this circuit (i.e. the contribution coming from each of the eigenvalues).

4. RLC filter bandwidth

Real inductors have series resistance, so when we try to make a CL (Capacitor - Inductor) filter, it is really a C(L series R) filter. Suppose that your goal is to sketch the Bode plot of the CL filter. This is easy for frequencies far away from the natural frequency $\omega_n = 1/\sqrt{LC}$ because it looks capacitive at low frequency and inductive at high frequency. The question is what does it look like near ω_n? This is what we are going to explore in this question.



(a) **Write down the impedance of a series RLC circuit in the form $Z_{RLC}(\omega) = A(\omega) + jX(\omega)$, where $X(\omega)$ is a real valued function of ω.**

(b) **Show that at $\omega = \omega_n$ that $X(\omega_n) = 0$ and that $Z_{RLC}(\omega_n) = R$. Show that the impedance of the capacitor and inductor are equal to $-jQR$ and $+jQR$ at this frequency.** Recall that for RLC circuits, the quality factor Q is defined to be the dimensionless (unitless) quantity $\frac{1}{R}\sqrt{\frac{L}{C}} = \frac{\omega_n L}{R} = \frac{1}{\omega_n RC}$. (This problem will help you understand why this is a reasonable quantity to define.)

(c) **What is $Z_{RLC}(\frac{\omega_n}{2})$? Show that $jX(\frac{\omega_n}{2}) = \frac{3}{4}Z_C(\frac{\omega_n}{2})$.**

(d) **What is $Z_{RLC}(2\omega_n)$? Show that $jX(2\omega_n) = \frac{3}{4}Z_L(2\omega_n)$.**

(e) In most LC and CL filters, we are interested in the bandwidth of the filter, which is the region in which the magnitude of the impedance does not change by more than a factor of $\sqrt{2}$ from its minimum value. Notice that the real part of the impedance Z_{RLC} is not changing with frequency and stays at R. What we care about is the frequency range where the imaginary part of the impedance does not dominate and instead stays in the range from $-jR$ to $+jR$.

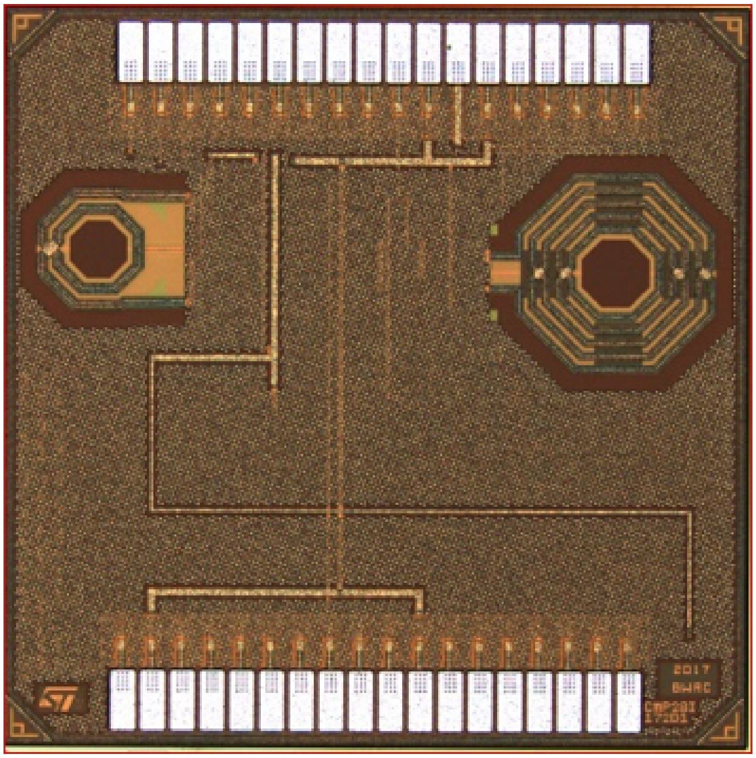
Write an expression for $X(\omega_n + \delta\omega)$, where $\delta\omega$ is a variable shift from ω_n. Find the values of $\delta\omega$ which give $X(\omega_n + \delta_1\omega) = -R$, and $X(\omega_n + \delta_2\omega) = +R$. These define the bandwidth of the filter. You may use the approximation that $\frac{1}{1+x} \approx 1 - x$ if $x << 1$.

(f) **What is the phase of the series impedance, $\angle Z_{RLC}(\omega)$, at the edges of the filter bandwidth, where $Z_{RLC}(\omega) = R - jR$ and where $Z_{RLC}(\omega) = R + jR$?**

(g) The bandwidth of the filter is the difference between the two frequencies at which $X(\omega_n + \delta_2\omega) = +R$ and $X(\omega_n + \delta_1\omega) = -R$. **Show that the bandwidth of the filter is approximately $\frac{\omega_n}{Q}$.** This bandwidth is approximate because of the $\frac{1}{1+x} \approx 1 - x$ if approximation where $|x| \ll 1$.

5. LC Tank

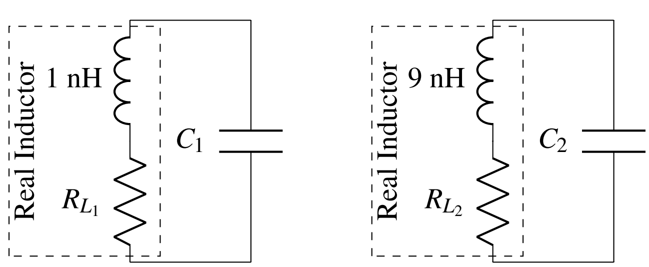
The chip below was designed in EE194 in Spring 2017. It is 1.1 mm on a side, and contains a 32 bit RISC-V microprocessor, 64kB RAM, and a 2.4 GHz Bluetooth Low Energy radio for communicating with cell phones.



There are two inductors on the chip. The inductor on the right is part of an impedance-matching circuit for the antenna interface. We will refer to this as the *antenna interface inductor*. Its inductance is 9 nH and the filter it is a part of has a Q of 8 at 2.4 GHz.

The inductor on the left has an inductance of 1nH, and participates in an LC filter with a Q of 15 at 4.8 GHz, and this LC filter is the "local oscillator" at the core of the radio's transmit- and receive-circuits. We will call this the *local oscillator inductor*.

There is a capacitor in parallel with each of the two inductors, which make resonators.



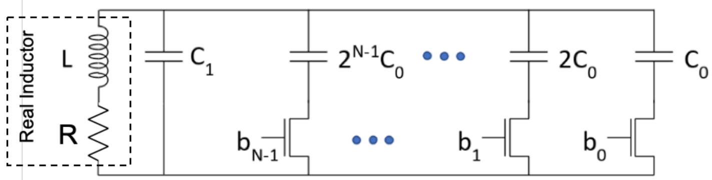
We are including the resistance that is present in real inductors in our model because our Q is not infinite. These circuit diagrams are meant to be representative of the two resonators described, which are a small part of the overall circuit seen in the chip above.

(a) **For the antenna interface inductor, what is the capacitance (to one significant figure) needed to make it resonant at 2.4 GHz?**

(b) **For the local oscillator inductor on the left, what capacitance (to four significant figures) is needed to make it resonant at 4.8 GHz? How about at 5 GHz?**

(c) **What is the impedance of the parallel LC tank (the local oscillator) at 4.8GHz?**
(HINT: We didn't give you the implicit resistance. What did we give you? Think about what the earlier problem taught you about the nature of Q.)

(d) Assume that the microprocessor with N bits can control a capacitor array that can give any capacitance between C_0 and $(2^{N-1})C_0$, shown below, for the local oscillator. (The capacitance in this case is controlled by a digital binary signal that turns transistors "on" or "off", which effectively selects the capacitors that are included in the array at any time.) This capacitor array is placed in parallel with a fixed capacitor C_1.
Our goal is to have the microprocessor set the frequency to any value between 4.8 GHz and 5 GHz with frequency error of no more than 100 kHz. **What value of C_1 should be used? What is the minimum number of bits N that is necessary to achieve the tunability goal? What is the best value for C_0?**



6. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

7. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **Who did you work on this homework with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

(d) **Roughly how many total hours did you work on this homework?**

8. Lecture notes and feedback

Staying up to date with lectures is an important part of the learning process in this course. This question is worth as much as the rest of the homework. Fortunately, it is also really easy.

(a) **Please attach your notes (handwritten or typed is fine) for the lecture from the Tuesday of the week before this HW is due *to the separate Gradescope assignment*.**

(b) **What did you think was the most important lesson of the Tuesday lecture?**

(c) **Please attach your notes (handwritten or typed is fine) for the lecture from the Thursday of the week before this HW is due *to the separate Gradescope assignment*.**

(d) **What did you think was the most important lesson of the Thursday lecture?**

(e) **Do you have any feedback on these lectures? How could they be improved in the future to better support students taking 16B?**

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