EECS 16B Designing Information Devices and Systems II
Spring 2019 A. Sahai, J. Roychowdhury, K. Pister **Homework 6**

This homework is due on Wednesday, March 13, 2019, at 11:59PM.
Self-grades are due on Saturday, March 16, 2019, at 11:59PM.

1. **Lecture notes and feedback**

   Staying up to date with lectures is an important part of the learning process in this course. This question is worth as much as the rest of the homework. Fortunately, it is also really easy.

   (a) **Please attach your notes (handwritten or typed is fine) for the lecture from the Tuesday of the week before this HW is due *to the separate Gradescope assignment*.**

   (b) **What did you think was the most important lesson of the Tuesday lecture?**

   (c) **Please attach your notes (handwritten or typed is fine) for the lecture from the Thursday of the week before this HW is due *to the separate Gradescope assignment*.**

   (d) **What did you think was the most important lesson of the Thursday lecture?**

   (e) **Do you have any feedback on these lectures? How could they be improved in the future to better support students taking 16B?**

2. **Analyzing A Microphone Board Circuit**

   In this problem, we will work up to analyzing a simplified version of the mic board circuit. In lab, we will address the minor differences between the final circuit in this problem and the actual mic board circuit. Recall that you'll be using the microphone to control your robot car.

   The microphone can be modeled as a frequency-dependent current source, $I_{MIC} = k\sin(\omega t) + I_{DC}$, where I_{MIC} is the current generated by the mic (which flows from VDD to VSS), I_{DC} is some constant current, k is the ratio that converts the force exerted by soundwaves on the mix's diaphragm to into current, and ω is the audio signal's frequency (in $\frac{\text{rad}}{\text{s}}$). VDD and VSS are 5 V and -5 V, respectively.

   (a) **DC Analysis** Assume for now that $k = 0$ (so that we can examine just the "DC" response of the circuit), **find $V_{\mathbf{OUT}}$ in terms of I_{DC}, R_1, R_2, and R_3** (Hint: You do not need to worry about V_{ss} in your calculations).

   (b) Now, let's include the sinusoidal part of I_{MIC} as well. We can model this situation as shown below, with I_{MIC} split into two current sources so that we can analyze the whole circuit using superposition. Let $I_{\text{AC}} = k\sin(\omega t)$. **Find and plot the function $V_{\mathbf{OUT}}(t)$.**

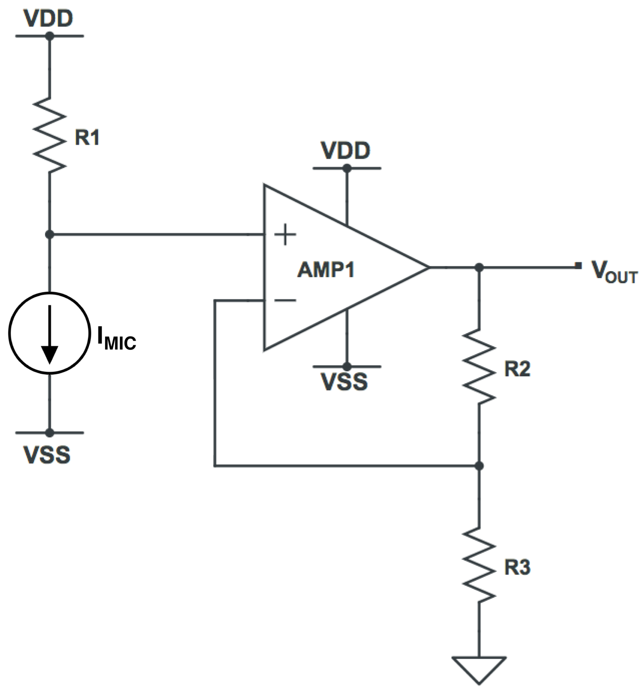


Figure 1: Step 1. The microphone is modeled as a DC current source.

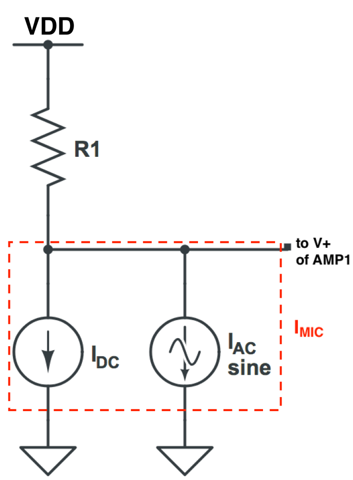


Figure 2: Step 2. The microphone is modeled as the superposition of a a DC and a sinusoidal ("AC") current source.

(c) Given that $V_{DD} = 5\,\text{V}$, $V_{SS} = -5\,\text{V}$, $R_1 = 10\,\text{k}\Omega$, and $I_{\text{DC}} = 10\,\mu\text{A}$, find the maximum value of the gain G of the noninverting amplifier circuit for which the op-amp would not need to produce voltages greater than VDD or less than VSS (i.e, **find the maximum gain G we can use without causing the op-amp to clip**).

(d) We have modified the circuit as shown below to include a high-pass filter so that the term related to I_{DC} is removed before we apply gain to the signal. **Provide a symbolic expression for V_{OUT} given that that $\text{VDD}_0 = 5\,\text{V}$, $\text{VSS}_0 = -5\,\text{V}$, $\text{VDD}_1 = 3.3\,\text{V}$, $\text{VSS}_1 = 0\,\text{V}$. Show your work.**

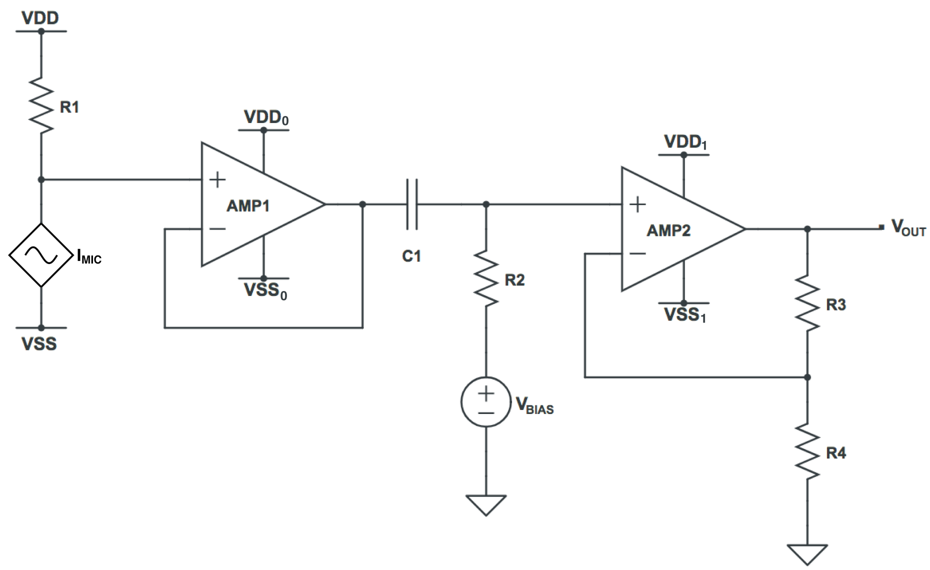


Figure 3: Step 3. Approaching the real mic board circuit. The microphone is still modeled as the superposition of a a DC and a sinusoidal ("AC") current source.

(e) We would now like to choose V_{BIAS} so that we can get as much gain G out of the non-inverting amplifier circuit (AMP2) as possible without causing AMP2 to clip (i.e, the output of AMP2 must stay between 0V and 3.3V). **What value of V_{BIAS} will achieve this goal? If $k = 10^{-5}$ and $R_1 = 10\,\text{k}\Omega$, what is the maximum value of G you can use without having AMP2 clip?**

3. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

$$\vec{x}(t+1) = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$.

(a) **Is this system controllable?**

(b) **Is this discrete-time linear system stable on its own?**

(c) Suppose we use state feedback of the form $u(t) = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}(t)$

Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

(d) Now suppose we've got a seemingly different system described by the controlled scalar difference equation $z(t+1) = z(t) + 2z(t-1) + u(t)$. **Write down the above system's representation in the following matrix form:**

$$\vec{z}(t+1) = A_z \vec{z}(t) + B_z u(t). \tag{1}$$

Please specify what the vector $\vec{z}(t)$ consists of as well as the matrix A_z and the vector B_z.

(HINT: Just as "state" in the case of continuous time refers to anything that has a derivative taken in the system of differential equations, for discrete time systems, the concept of state refers to memory. What, besides the current input, must you remember about the past/present to be able to figure out the future? In this case, you must know both $z(t)$ and $z(t-l)$.)

(e) We will now show how the initial matrix representation for $\vec{x}(t)$ can be converted to the canonical form for $\vec{z}(t)$ using a change of basis. Suppose we do a transformation of the coordinates of the state $\vec{x}(t)$ to $\vec{z}(t) = P\vec{x}(t)$. **Write down the state-transition matrices of $\vec{z}(t)$ in terms of the state transition matrices of $\vec{x}(t)$, i.e., express A_z and B_z in terms of A, B, and P.** For $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, confirm that the resulting state space representation of the behavior of $\vec{z}(t)$ is indeed the same as the previous part (i.e. we get the same A_z, B_z).

(f) For the previous part, **Design a feedback $\begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix}$ to place the closed-loop eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Confirm that $\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P$.**

(g) **(BONUS, but in scope)** Here, we gave you the P matrix. **How would you have come up with the P matrix on your own?** (Hint: start with the second column of P and ask where it might have come from. Then, is there a relationship between the coefficients of the difference equation in part (d) to the polynomial whose roots you need to find in part (b)?)

By following this technique, any controllable discrete-time system can be converted to the "controllable canonical form" shown in part (d) by finding the right change of coordinates.

(h) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrix A and B from (a) for the system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) + w(t)$$

into the Jupyter notebook and use the random input $w(t)$ as the disturbance introduced into the state equation. Observe how the norm of $\vec{x}(t)$ evolves over time for the given A. **What do you see happening to the norm of the state?**

(i) Add the feedback computed in part (c) to the system in the notebook and **explain how the norm of the state changes.**

(j) Now we evaluate a system described by the following scalar system $z(t+1) = az(t) + u(t) + w(t)$ in the Jupyter notebook. Consider two values of a, one case with $a > 1$ and one with $a < 1$, to observe the difference in the evolution of $|z(t)|$ for the same error as part (g). **Describe the differences between the two.**

(k) Suppose that the disturbance is actually coming from observation noise. We assume $y(t) = z(t) + w(t)$ where $w(t)$ is some random noise. Add a state feedback $u(t) = ky(t)$ to the system so that the resulting closed loop system is described by $z(t+1) = (a+k)z(t) + kw(t)$. Say we know $a = -1.25$. **For what values of k's will the result be bounded. Confirm with the norm of the closed loop system.**

(l) **Is it advisable to have $a+k$ close to zero if we want to minimize the magnitudes of the state $x(t)$? How does the effect of the noise in the observation influence this?** Assign values of k close to $-a$ to see the effect in the Jupyter notebook. Compare to values that are smaller, but still keep it stable.

4. Tracking a Desired Trajectory in Continuous Time

The treatment in 16B so far has treated closed-loop control as being about holding a system steady at some desired operating point, which was designated as zero in a linear model. This control used the actual current state to apply a control signal designed to bring the state to zero. Meanwhile, the idea of controllability itself was more general and allowed us to make an open-loop trajectory that went pretty much anywhere. This problem is about combining these two ideas together to make feedback control more practical — how can we get a system to more-or-less closely follow a desired trajectory, even though it might not start exactly where we wanted to start and in principle could be buffeted by small disturbances throughout.

The key conceptual idea is to realize that we can change coordinates in a time-varying way so that "zero" is the desired "open-loop" trajectory.

In this question, we will also see that everything that you have learned to do closed-loop control in discrete-time can also be used to do closed-loop control in continuous time.

Now, consider the specific 2-dimensional system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{2}$$

where $u(t)$ is a scalar valued continuous control input.

(a) **Would the given system be controllable if we viewed the $A, \vec{b}$ as the parameters of a discrete-time system?**

(b) Now, suppose we started at $\vec{x}(0) = \vec{0}$ and had a nominal control signal $u_n(t)$ that would make the system follow the desired trajectory $\vec{x}_n(t)$ that satisfies (2) together with $u_n(t)$.

Change variables using $\vec{x}(t) = \vec{x}_n(t) + \vec{v}(t)$ and $u(t) = u_n(t) + u_v(t)$ and write out what (2) implies for the evolution of the trajectory deviation $\vec{v}(t)$ as a function of the control deviation $u_v(t)$.

Now, add a bounded disturbance term $\vec{w}(t)$ to the original state evolution in (2) and let's see if we can absorb that entirely within an evolution equation for $\vec{v}(t)$ you found above. Write out the resulting equation for the dynamics as:

$$\frac{d}{dt}\vec{v}(t) = A_v\vec{v}(t) + \vec{b}_v u_v(t) + \vec{w}(t) \tag{3}$$

What are A_v and $\vec{b}_v$?

(c) Based on what you have found above, how will the system behave over time? **If there is some small distubance, will we end up following the intended trajectory $\vec{x}_n(t)$ closely if we just apply the control $u_n(t)$ to the original system?**

Now, we want to apply state feedback control to the system to get it to more or less follow the desired trajectory.

(d) Just looking at the $\vec{v}(t), u_v(t)$ system, **how would you apply state-feedback to choose $u_v(t)$ as a function of $\vec{v}(t)$ that would place both the eigenvalues of the closed-loop $\vec{v}(t)$ system at -10.**

(e) Based on what you did in the previous parts, and given access to the desired trajectory $\vec{x}_n(t)$, the nominal controls $u_n(t)$, and the actual measurement of the state $\vec{x}(t)$, **come up with a way to do feedback control that will keep the trajectory staying close to the desired trajectory no matter what the small bounded disturbance $\vec{w}(t)$ does.**

5. Solar Panel

A particular solar panel's IV curves (labeled by irradiance) are given by:

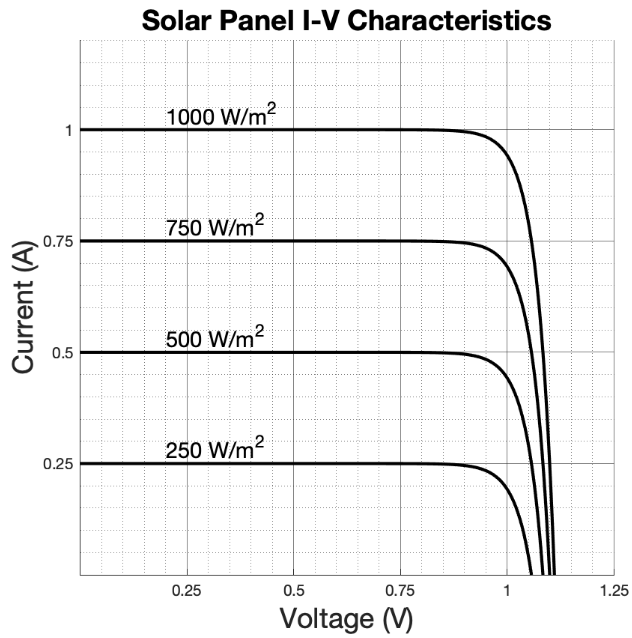
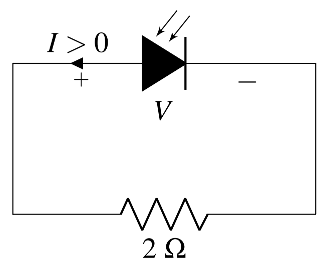


Figure 4: Solar Panel IV Curve

Your solar panel is connected to a load with resistance $R = 2\ \Omega$. The circuit diagram is given below, where the solar panel is depicted as a single pn-junction diode. Remember that current appears to flow "backward" out of the solar cell diode because it is *producing* power, rather than dissipating it.



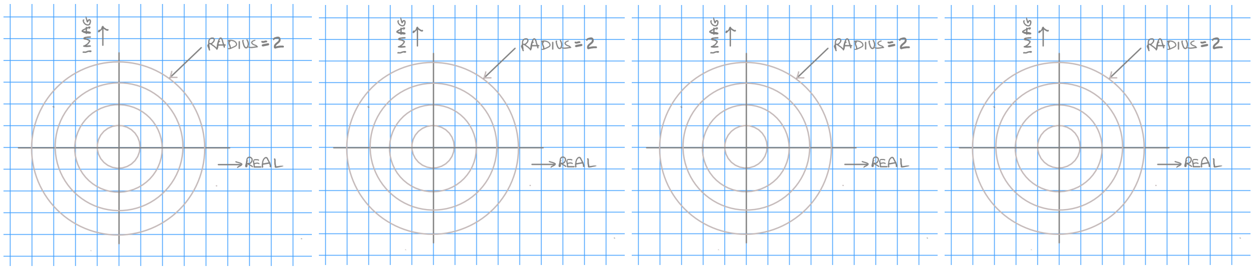
(a) **Draw the resistor's IV curve on the plot which includes the solar panel IV curves above.** The intersection between your resistance curve and a particular IV curve tells you where the solar panel will end up operating given a particular irradiance.

(b) **If the irradiance were 250 W/m^2, what would the power dissipated across the resistor be?**

6. Phasors

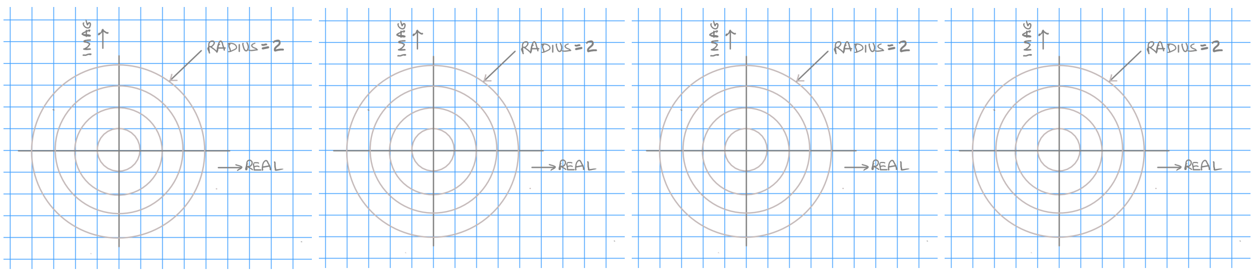
(a) Consider a resistor ($R = 1.5\Omega$), a capacitor ($C = 1$F), and an inductor ($L = 1$H) connected in series. **Give expressions for the impedances of Z_R, Z_C, Z_L for each of these elements as a function of the angular frequency ω.**

(b) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = \frac{1}{2}$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.



a $Z_R(@\omega = 0.5)$ b $Z_C(@\omega = 0.5)$ c $Z_L(@\omega = 0.5)$ d $Z_{total}(@\omega = 0.5)$

Figure 5: Impedances at $\omega = 0.5$.

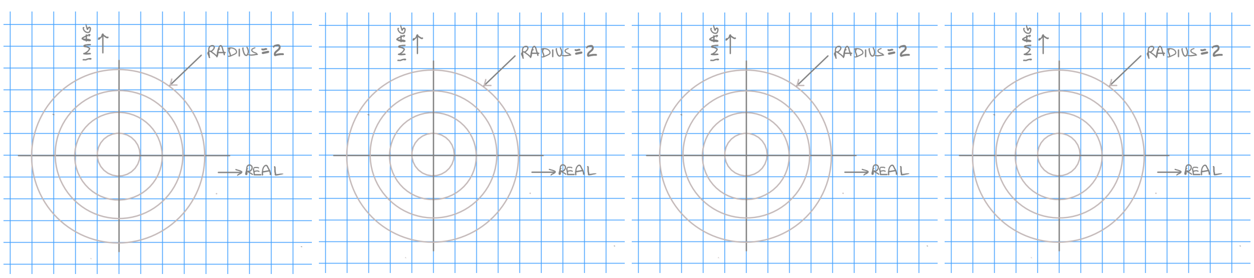


a $Z_R(@\omega = 1)$ b $Z_C(@\omega = 1)$ c $Z_L(@\omega = 1)$ d $Z_{total}(@\omega = 1)$

Figure 6: Impedances at $\omega = 1$.

(c) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 1$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.

(d) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 2$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.



a $Z_R(@\omega = 2)$ b $Z_C(@\omega = 2)$ c $Z_L(@\omega = 2)$ d $Z_{total}(@\omega = 2)$

Figure 7: Impedances at $\omega = 2$.

(e) **For the previous series combination of RLC elements, what is the "natural frequency" ω_n where the series impedance is purely real?**

(f) Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{4}$$

where for this problem, we assume that $u(t)$ has a phasor representation $\widetilde{U}$. In other words, $u(t) =$

$\widetilde{U}e^{+j\omega t} + \overline{\widetilde{U}}e^{-j\omega t}$. Suppose further that all the eigenvalues of A are such that any impact of an initial condition has completely died out by now. (i.e. the system is in steady-state.)

Assume that the vector solution $\vec{x}(t)$ to the system of differential equations (4) can also be written in phasor form as

$$\vec{x}(t) = \widetilde{\vec{X}}e^{+j\omega t} + \overline{\widetilde{\vec{X}}}e^{-j\omega t}. \tag{5}$$

Derive an expression for $\widetilde{\vec{X}}$ involving $A, \vec{b}$, $j\omega$, $\widetilde{U}$, and the identity matrix I.

(HINT: Plug (5) *into* (4) *and simplify, using the rules of differentiation and grouping terms by which exponential $e^{\pm j\omega t}$ they multiply.)*

7. Low-pass Filter

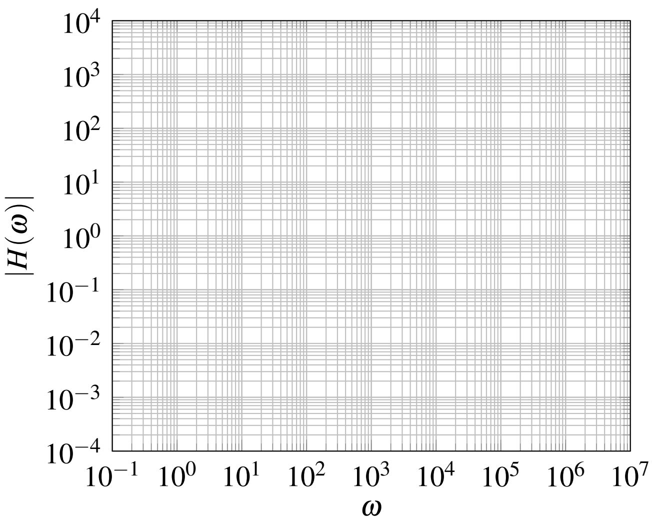
You have a 1 kΩ resistor and a 1 μF capacitor wired up as a low-pass filter.

(a) **Draw the filter, labeling the input node, output node, and ground.**

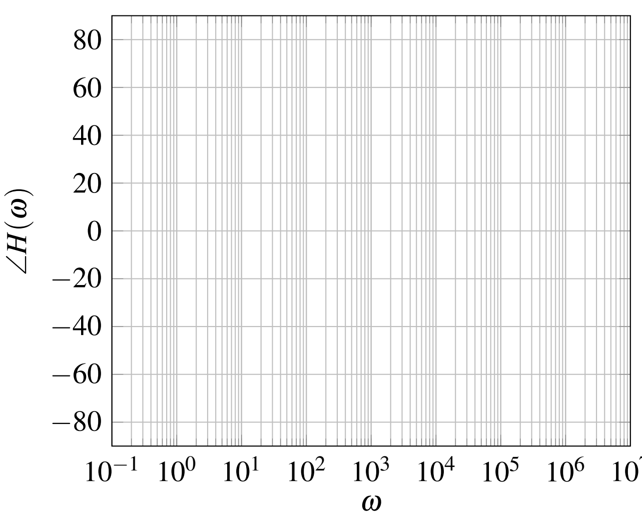
(b) **Write down the transfer function of the filter, $H(\omega)$.** Be sure to use the given values for the components.

(c) **Draw a straight-line approximation to the Bode plot (both magnitude and phase) of the filter on the graph paper below.**

Log-log plot of transfer function magnitude



Semi-log plot of transfer function phase



(d) **Annotate your Bode plot with three circles, each representing where the straight line approximation has its worst errors.** One circle should be on the magnitude plot, and two should be on the phase plot and corresponds to an abolute error. **Label each circle with the error at that point** (multiplicative error in the case of the magnitude plot, and absolute error in terms of the phase plot). For the phase plot, feel free to use trigonometric functions if you want.

(e) **Write an exact expression for the *magnitude* of $H(\omega = 10^6)$, and give an approximate numerical answer.**

(f) **Write an exact expression for the *phase* of $H(\omega = 1)$, and give an approximate numerical answer.**

(g) **Write down an expression for the time-domain output waveform $V_{out}(t)$ of this filter if the input voltage is $V(t) = 1\sin(1000t)$ V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

8. What Is The Use of a Ferrite Bead?

You've probably noticed how most laptop chargers have a short, thick section near the end that plugs into the laptop.

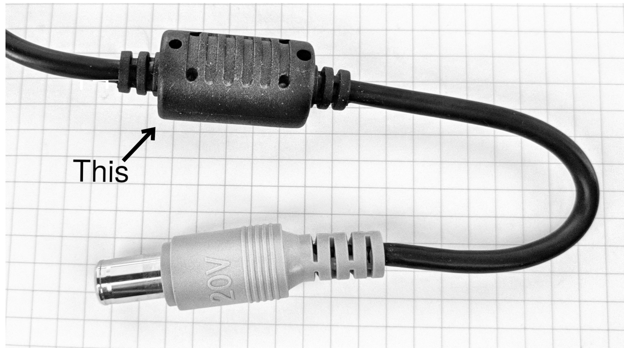


Figure 8: The short, thick section in question.

It's called a *ferrite bead*. It's a small shell of magnetic material called *ferrite*, and it makes the section of the wire that it surounds into **more** of an inductor (all wires have some inherent inductance, and in long wires, this is not negligible). Its purpose is to help filter out power supply noise. In circuit terms, the setup looks like this:

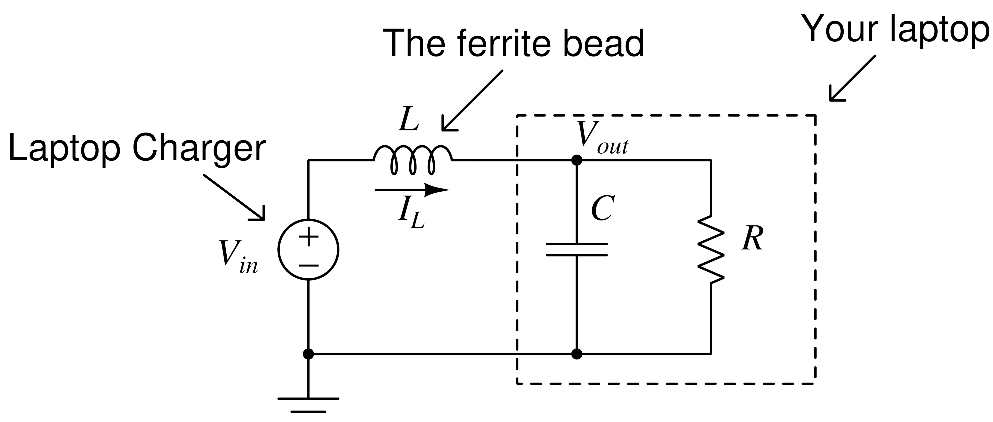


Figure 9: The ferrite bead in its natural habitat. L includes the wire inductance as well.

Here, V_{in} is the voltage produced by the charger, and V_{out} is the voltage that reaches the laptop. The resistor models the power consumption of the laptop, and the capacitor is part of an internal power supply filter.

In power supply filter design, the *time-domain* behavior of the filter is just as important as the *phasor-domain* behavior. Because we have more time in HW than on the exam, here you can also examine the phasor-domain behavior.

(a) Using $x_1(t) = I_L(t)$ and $x_2(t) = V_{out}(t)$ as state variables, **construct a matrix differential equation**

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}V_{in}(t) \tag{6}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. i.e. What are A and $\vec{b}$ for this circuit?

(b) **What are the eigenvalues of the matrix A you found in the previous part?** Your answer should be in terms of R, L, and C.

(c) The reason that we care about the time-domain behavior of this circuit is that *we must prevent resonance*. Although LC resonance is often useful, in a power supply it must be avoided to prevent dangerous high-voltage oscillations from occurring.

How can we tell if the filter we design will resonate or not? Well, any imaginary component to the eigenvalues can induce oscillation. Using the eigenvalues you found in the previous part, **find the smallest value for L such that the system will not oscillate. Your answer should be in terms of R and C.**

(d) Suppose $R = 4\,\Omega$, $C = 10\,\mu\text{F}$. Using these values, **choose the smallest value for L that does not allow for LC resonance.**

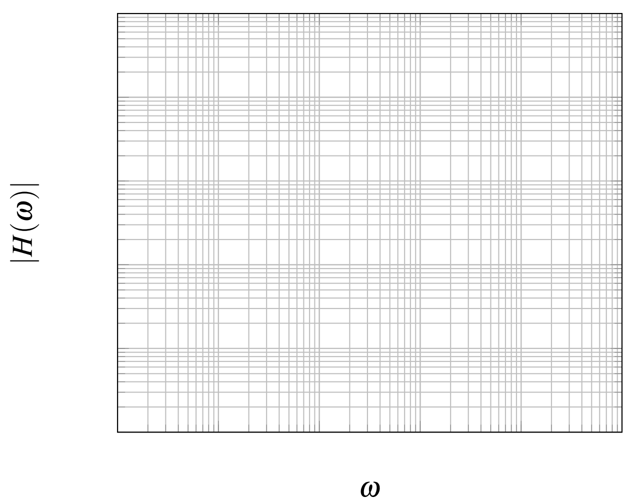
(e) Now that we have chosen the time-domain behavior that we wanted, we'll need to see how the circuit behaves in the frequency domain. Treating the source voltage V_{in} as a phasor $\widetilde{V_{in}}$, and treating the inductor and capacitor as impedances, **Determine the transfer function** $H(\omega) = \frac{\widetilde{V_{out}}}{\widetilde{V_{in}}}$, in terms of R, L, and C.

(f) Using the numerical values for R and C that we specified, and the numerical value for the minimal L that you found, **rewrite the transfer function as a product of two poles**. It will turn out that it can be written in the form

$$H(\omega) = \frac{1}{(1 + j\omega/\omega_{p1})(1 + j\omega/\omega_{p2})}, \tag{7}$$

where ω_{p1} and ω_{p2} are some values you will determine, which may or may not be equal to each other.

(g) Now that you have the transfer function written as the product form above, **Draw the *magnitude Bode plot* for this transfer function** on the following blank log-log plot.



(h) The reason we need to do frequency analysis on this circuit is to judge whether or not it's any good at rejecting high-frequency noise. Suppose that the power supply produces some interfering signal at 10 Mrad/s that we want to remove. For our purposes, we'd like for this interference to be attenuated by a factor of at least a 1000 by this filter. Using the given values for R and C, and the value of L you chose from the previous parts, **Does the filter satisfy this design requirement?**

You may use either the transfer function or the Bode plot to answer this question, whichever you prefer. An exact value for the attenuation isn't necessary, just enough to answer yes or no. There's no shame if the answer is no, by the way: sometimes, the first design you try won't work.

9. Transistor Switch Model

You have two CMOS inverters made from NMOS and PMOS devices. Both NMOS and PMOS devices have an "on resistance" of $R_{on} = 1\ \text{k}\Omega$, and each has a gate capacitance (input capacitance) of $C = 1$ fF (femto-Farads = 10^{-15}). We assume the "off resistance" (the resistance when the transistor is off) is infinite (*i.e.*, the transistor acts as an open circuit when off). The supply voltage V_{DD} is 1V. The two inverters are connected in series, with the output of the first inverter driving the input of the second inverter.

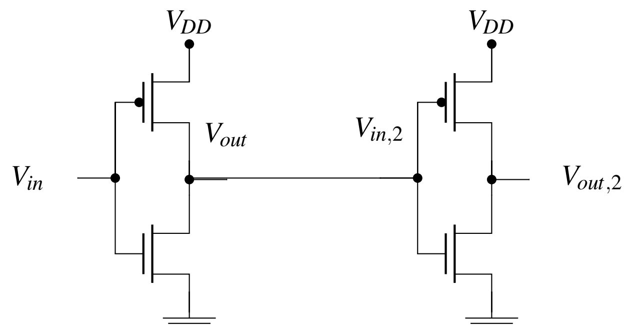


Figure 10: CMOS Inverter

(a) Assume the input to the first inverter has been low ($V_{in} = 0$ V) for a long time, and then switches at time $t = 0$ to high ($V_{in} = V_{DD}$). **Draw a simple RC circuit and write a differential equation describing the output voltage of the first inverter for time $t \geq 0$.** Don't forget that the second inverter is "loading" the output of the first inverter — you need to think about both of them.

(b) **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the initial slope, (3) the asymptotic value, and (4) the time that it takes for the voltage to decay to roughly 1/3 of its initial value.**

(c) A long time later, the input to the first inverter switches low again. **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the initial slope, and (3) the asymptotic value.**

(d) For each complete input cycle described in the two steps above, **how much charge is pulled out of the power supply?** Give both a symbolic and numerical answer.

10. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

11. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **Who did you work on this homework with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

(d) **Roughly how many total hours did you work on this homework?**

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