EECS 16B Designing Information Devices and Systems II
Spring 2019 A. Sahai, J. Roychowdhury, K. Pister Homework 7

This homework is due on Wednesday, March 20, 2019, at 11:59PM.
Self-grades are due on Saturday, March 23, 2019, at 11:59PM.

1. Discrete systems and their orbits

The concept of controllability exists to tell us whether or not a system can be eventually driven from any initial condition to any desired state given no disturbances, perfect knowledge of the system, and knowledge of the initial state.

For a discrete-time system with n-dimensional state $\vec{x}$ driven by a scalar input

$$\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i) \tag{1}$$

this can be checked by seeing whether the controllability matrix

$$C = \begin{bmatrix} \vec{b} & A\vec{b} & A^2\vec{b} & \cdots & A^{n-1}\vec{b} \end{bmatrix} \tag{2}$$

has a range (span of the columns) that encompasses all of $\mathbb{R}^n$. In other words, $\text{span}(C) = \mathbb{R}^n$. If it does span the whole space, then the system (1) is controllable. If it does not, then the system is not controllable.

Everything seems to hinge on the "orbit" that the vector $\vec{b}$ takes as it repeatedly encounters A. Does this orbit confine itself to a subspace, or does it explore the whole space? (Formally, this orbit is the infinite sequence $\vec{b}, A\vec{b}, A^2\vec{b}, \cdots$)

For almost the entire rest of the problem, let us assume that the square matrix A has n distinct and real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding nontrivial eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

(a) Suppose that we choose $\vec{b} = \alpha\vec{v}_i$ for some eigenvector $\vec{v}_i$. **Show that the rank of C will be 1.** i.e. Show that the subspace spanned by the orbit of $\vec{b}$ through A will be one dimensional.

(b) If $\vec{b} = \vec{v}_i$ as above, **would the system be controllable if $n > 1$?**

(c) If $\vec{b} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$, then **show that $A^2\vec{b} = \beta_1 A\vec{b} + \beta_0\vec{b}$ for some choice of β_1 and β_0.**

(d) In the previous part, if all the $\alpha_i \neq 0$, **do the coefficients β_i depend on the exact value of the α_i?**

(e) Consequently, in the previous part, if $n > 2$, **would the system be controllable if the $\vec{b}$ was a linear combination of only two eigenvectors?**

(f) Now consider a general square matrix A (not necessarily with distinct eigenvectors, etc.) with a specific $\vec{b}$ such that the system defined by the pair $(A, \vec{b})$ is controllable. For this specific vector $\vec{b}$, there exist $\{\beta_i\}_{i=0}^{n-1}$ so that it satisfies:

$$A^n\vec{b} = \sum_{i=0}^{n-1} \beta_i A^i \vec{b} \tag{3}$$

Show that for all $j > 0$, the vector $\vec{w}_j = A^j\vec{b}$, also satisfies $A^n\vec{w}_j = \sum_{i=0}^{n-1} \beta_i A^i \vec{w}_j$.
(HINT: Multiply both sides of an equation by something.)

(g) Suppose your general square matrix A above has a specific $\vec{b}$ that satisfies (3) above and such that the system defined by the pair $(A, \vec{b})$ is controllable. **Use the previous part to show that in fact, $A^n - \beta_{n-1}A^{n-1} - \beta_{n-2}A^{n-2} - \cdots - \beta_1 A - \beta_0 I$ is the matrix of all zeros.**

(HINT: The previous part might provide you with a very convenient basis to use. Then remember that the signature of the zero matrix is that no matter what it multiplies, it returns zero.)

Recall that in lecture, we argued why this specific polynomial $(\lambda^n - \beta_{n-1}\lambda^{n-1} - \cdots - \beta_1\lambda - \beta_0)$ must be the characteristic polynomial $(\det(\lambda I - A))$ of the matrix A. So, this argument shows that these kinds of matrices must satisfy their own characteristic polynomials. (It is also easy to see this for diagonalizable matrices A, but this example shows that it holds more generally for controllable matrices.)

This argument can actually be used to extend to all square matrices, even those that don't have a full complement of linearly independent eigenvectors and aren't controllable with a single scalar input. When extended all the way, it is called the Cayley-Hamilton theorem. It shows that a square matrix satisfies its own characteristic polynomial — no matter what.

2. Controllable Canonical Form and Eigenvalue Placement

Consider a linear discrete time system below ($\vec{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $\vec{b} \in \mathbb{R}^n$).

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t)$$

If the system is *controllable*, then there exists a transformation $\vec{z} = T\vec{x}$ (where T is an invertible $n \times n$ matrix) such that in the transformed coordinates, the system is in *controllable canonical form*, which is given by

$$\vec{z}(t+1) = \widetilde{A}\vec{z}(t) + \widetilde{\vec{b}}u(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

Here, $\widetilde{A} = TAT^{-1}$ and $\widetilde{\vec{b}} = T\vec{b}$.

The characteristic polynomials of the matrices A and $\widetilde{A}$ are the same and given by

$$\det(\lambda I - A) = \det(\lambda I - \widetilde{A}) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \cdots - a_0. \tag{4}$$

(a) **Directly show that A and $\widetilde{A}$ have the same eigenvalues.**

(*Hint: let $\vec{v}$ be an eigenvector of A; use $T\vec{v}$ for $\widetilde{A}$*)

(b) Let the controllability matrices (what we use to check controllability) C and $\widetilde{C}$ be $C = \begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{n-1}\vec{b} \end{bmatrix}$ and $\widetilde{C} = \begin{bmatrix} \widetilde{\vec{b}} & \widetilde{A}\widetilde{\vec{b}} & \ldots & \widetilde{A}^{n-1}\widetilde{\vec{b}} \end{bmatrix}$, respectively. **Show that $\widetilde{C} = TC$.** Notice that this is equivalent to $T = \widetilde{C}C^{-1}$ since all of these matrices are invertible.

(HINT: Write out $\widetilde{C}$ and substitute in TAT^{-1} for $\widetilde{A}$ and $T\vec{b}$ for $\widetilde{\vec{b}}$. Then, cancel $T^{-1}T$ terms where you can. Then, remember what you were taught in 16A about what matrix multiplication means in terms of matrix vector multiplication — i.e. multiplying DF is just a matrix whose i-th column is $D\vec{f}_i$ where $\vec{f}_i$ is the i-th column of F.)

(c) **Play with the included jupyter notebook and comment on the difficulty of setting controller gains by hand relative to using CCF.**

From this point onward, the parts of this problem are optional. Doing them provides extra practice in using CCF by hand. But feel free to skip if you think this is too easy and tedious.

Now, consider the specific system

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t) = \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u(t) \tag{5}$$

(d) (Optional, but in scope.) **Show that the system (5) is controllable.**

Since the system is controllable, there exists a transformation $\vec{z} = T\vec{x}$ such that

$$\vec{z}(t+1) = \widetilde{A}\vec{z}(t) + \widetilde{\vec{b}}u(t) = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{6}$$

The characteristic polynomials of the matrices A and $\widetilde{A}$ are the same.

(e) (Optional, but in scope.) **Compute the matrix $\widetilde{A}$.**

(f) (Optional, but in scope.) **Compute the controllability matrices $C = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix}$ and $\widetilde{C} = \begin{bmatrix} \widetilde{\vec{b}} & \widetilde{A}\widetilde{\vec{b}} \end{bmatrix}$.**

(g) (Optional, but in scope.) **Compute the transformation matrix $T = \widetilde{C}C^{-1}$.**

(h) (Optional, but in scope.) **Show that the system (5) is *unstable* in open-loop.** (That is, when we do not apply closed-loop feedback.)

Now, we want to make the system *stable* by applying state feedback for the system in its original (5) and canonical (6) forms.

That is, let $u(t)$ be $u(t) = -\vec{k}^T\vec{z}(t) = \begin{bmatrix} -k_0 & -k_1 \end{bmatrix} \vec{z}(t)$. After applying state feedback, the systems (5) and (6) have the form

$$\vec{x}(t+1) = A_{cl}\vec{x}(t)$$
$$\vec{z}(t+1) = \widetilde{A}_{cl}\vec{z}(t)$$

(i) (Optional, but in scope.) **Compute A_{cl} and $\widetilde{A}_{cl}$ in terms of k_0 and k_1.**

(j) (Optional, but in scope.) **Compute $\vec{k}$ so that $\widetilde{A}_{cl}$ has eigenvalues $\lambda = \pm\frac{1}{2}$** (*Hint: use the characteristic polynomial (4).*)

(k) (Optional, but in scope.) Using the $\vec{k}$ you derived in the previous part, **show that A_{cl} also has eigenvalues $\lambda = \pm\frac{1}{2}$ by explicit calculation.** (Feel free to use numpy to do this.)

3. Single-dimensional linearization

This is an exercise around linearization of a scalar system. The scalar nonlinear differential equation we have is

$$\frac{d}{dt}x(t) = \sin\big(x(t)\big) + u(t). \tag{7}$$

(a) The first thing we want to do is find equilibria (DC operating points) that this system can support. Suppose we want to investigate potential expansion points (x^*, u^*) with $u^* = 0$. **Sketch** $\sin(x^*)$ **for** $-4\pi \leq x^* \leq 4\pi$ **and intersect it with the horizontal line at** 0**.** This will show us the equilibria points, where $sin(x^*) + u^* = 0$.

(b) **Show that all** $x(t) = x_m^* = m\pi$ **satisfy** (7) **together with** $u^* = 0$**.**

Let us zoom in on two choices: $x_{-1}^* = -\pi$ and $x_0^* = 0$. (Looking at the sketch we made, these seem like representative points.)

(c) Linearize the system (7) around the equilibrium $(x_0^*, u^*) = (0, 0)$. **What is the resulting linearized scalar differential equation for** $x_\ell(t) = x(t) - x_0^* = x(t) - 0$**, involving** $u_\ell(t) = u(t) - u^* = u(t) - 0$**?**

(d) For the linearized approximate system model that you found in the previous part, what happens if we try to discretize time to intervals of duration Δ? Assume now we use a piecewise constant control input over duration Δ, that Δ is small relative to the ranges of controls applied, and that we sample the state x every Δ (that is, at every $t = n\Delta$, where n is an integer) as well. **Write out the resulting scalar discrete-time control system model.** This model is an approximation of what will happen if we actually applied a piecewise constant control input to the original nonlinear differential equation.

(e) **Is the (approximate) discrete-time system you found in the previous part stable or unstable?**

(f) Now linearize the system (7) around the equilibrium $(x_{-1}^*, u^*) = (-\pi, 0)$. **What is the resulting approximate scalar differential equation for** $x_\ell(t) = x(t) - (-\pi)$ **involving** $u_\ell(t) = u(t) - 0$**?**

(g) For the linearized system model that you found in the previous part, what happens if we try to discretize time to intervals of duration Δ? Assume now we use a piecewise constant control input over duration Δ, that Δ is small relative to the ranges of controls applied, and that we sample the state x every Δ (that is, at every $t = n\Delta$, where n is an integer) as well. **Write out the resulting scalar discrete-time control system.** This model is an approximation of what will happen if we actually applied a piecewise constant control input to the original nonlinear differential equation.

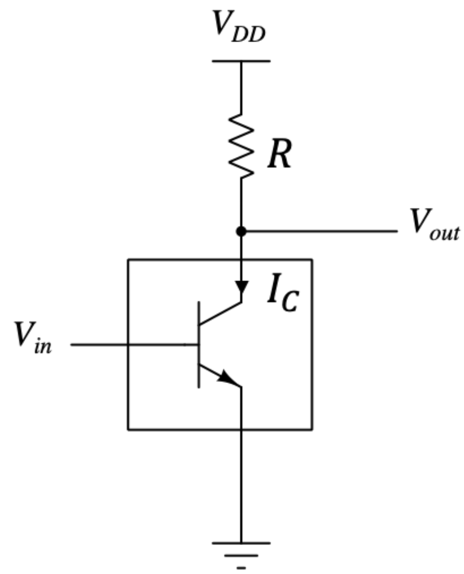
(h) **Is the (approximate) discrete-time system you found in the previous part stable or unstable?**

(i) Suppose for the two *discrete-time systems* above, we chose to apply a feedback law $u(t) = -k(x(t) - x^*)$. **for what range of** k **values, would the resulting linearized discrete-time systems be stable?** Your answer will depend on Δ.

4. Linearizing for understanding amplification

Linearization isn't just something that is important for control, robotics, machine learning, and optimization — it is one of the standard tools used across different areas, including thinking about circuits.

The circuit below is a voltage amplifier, where the element inside the box is a bipolar junction transistor (BJT).



The bipolar transistor in the circuit can be modeled quite accurately as a nonlinear, voltage-controlled current source, where the collector current I_C is given by

$$I_C(V_{in}) = I_S e^{\frac{V_{in}}{V_{TH}}} \tag{8}$$

where V_{TH} is the thermal voltage. We can assume $V_{TH} = 26$ mV at temperatures of 300K (close to room temperature).

With this amplifier, small variations in the input voltage V_{in} can turn into large variations in the output voltage V_{out} under the right conditions. We're going to investigate this amplification using linearization.

Let's consider the 2N3904 transistor, where the above expression for $I_C(V_{in})$ holds as long as $0.2\text{V} < V_{out} < 40\text{V}$, and $0.1\text{mA} < I_C < 10\text{mA}$.

(Note that the 2N3904 is a cheap transistor that people often use in personal projects. You can get them for 3 cents each if you buy in bulk.)

(a) **Write a symbolic expression for V_{out} as a function of I_C.**

(b) Now let's linearize I_C in the neighborhood of an input voltage V_{in}^* and a specific I_C^*. Assume that you have a found a particular pair of input voltage V_{in}^* and current I_C^* that satisfy the current equation (8).

We can look at nearby input voltages and see how much the current changes. We can write the linearized expression for the collector current around this point as:

$$I_C(V_{in}) = I_C(V_{in}^*) + \delta I_C \approx I_C^* + m(V_{in} - V_{in}^*) = I_C^* + m\, \delta V_{in} \tag{9}$$

where $\delta V_{in} = V_{in} - V_{in}^*$ is the change in input voltage and $\delta I_C = I_C - I_C^*$ is the change in collector current. **What is m here as a function of I_C^* and V_{TH}?**

(If you take EE105, you will learn that this m is called the transconductance, which is usually written g_m, and is the single most important parameter in most analog circuit designs.)

(HINT: First just find m by taking the appropriate derivative and using the chain rule as needed. Then leverage the special properties of the exponential function to express it in terms of the desired quantities.)

(c) We now have a linear relationship between small changes in current and voltage, $\delta I_C = m\,\delta V_{in}$ around a known solution (I_C^*, V_{in}^*). This is called a "bias point" in circuits terminology.

Going back to your equation from part (a), plug in your linearized equation for I_C. Define the appropriate V_{out}^* so that it makes sense to view $V_{out} = V_{out}^* + \delta V_{out}$ when we have $V_{in} = V_{in}^* + \delta V_{in}$, and **find the approximate linear relationship between δV_{out} and δV_{in}.**

The ratio $\frac{\delta V_{out}}{\delta V_{in}}$ is called the small-signal voltage gain of this amplifier around this bias point.

(d) Assuming that $V_{DD} = 10V$, $R = 1k\Omega$, and $I_C^* = 1mA$ when $V_{in}^* = 0.65V$, **what is the small-signal voltage gain $\frac{\delta V_{out}}{\delta V_{in}}$, between the input and the output around this bias point?** (one or two digits of precision is plenty)

(e) If $I_C^* = 9mA$ when $V_{in}^* = 0.7V$, **what is the small-signal voltage gain around this bias point?** (one or two digits is plenty)

This shows you how by appropriately biasing (choosing an operating point), we can adjust what our gain is for small signals. Although here, we just wanted to show you this as a simple application of linearization, these ideas are developed a lot further in 105, 140, and other courses to create things like op-amps and other analog information-processing systems.

5. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

6. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **Who did you work on this homework with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

(d) **Roughly how many total hours did you work on this homework?**

Contributors:

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