

# 1 What are Complex Numbers?

## 1.1 Introduction

Most students have some basic background in complex numbers ( $\mathbb{C}$ ) from high school. The purpose of this note is to solidify this understanding. With that, let's begin with the most basic definition:  $j = \sqrt{-1}$  (in most engineering disciplines, we will use  $j$ , so as not to confuse ourselves with current or the identity matrix  $I$ . Besides, numpy uses  $j$  as well). All complex numbers are of the form  $z = a + jb$ , where  $a$  is the **real part** and  $b$  is the **imaginary part**. This form is more commonly known as the **rectangular form**, and as we will see, addition in this form is very easy and akin to vector addition.

The **complex conjugate**<sup>1</sup> of  $z$ , represented by  $\bar{z}$  (and sometimes by  $z^*$ ), is defined as follows:

$$\bar{z} = \overline{(a + jb)} = a - jb \quad (1)$$

The **magnitude** of a complex number,  $z$ , is given by

$$|z| = \sqrt{a^2 + b^2} \quad (2)$$

and the **phase** is given by

$$\theta = \angle z = \text{atan2}(b, a) \quad (3)$$

Here,  $\text{atan2}(y, x)$  is a function<sup>2</sup> that returns the angle from the positive x-axis to the vector from the origin to the point  $(x, y)$ .

## 1.2 Basic Operations

Let's say we have two complex numbers  $z_1 = a_1 + jb_1$  and  $z_2 = a_2 + jb_2$ . As you may recall, addition is defined as follows:

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \quad (4)$$

This should look very similar to the addition of vectors in 2D. Multiplication is a little more complicated, but it behaves very similar to polynomial multiplication, except the indeterminate (i.e. the variable) is replaced by  $j$ , and we have  $j^2 = -1$ :

$$z_1 \times z_2 = (a_1 + jb_1) \times (a_2 + jb_2) \quad (5)$$

$$= a_1 * a_2 + jb_1 * a_2 + ja_1 * b_2 + j^2 b_1 * b_2 \quad (6)$$

$$= (a_1 * a_2 - b_1 * b_2) + j(a_1 * b_2 + a_2 * b_1) \quad (7)$$

<sup>1</sup> $\bar{z}$  is also used in digital logic to represent 'NOT' operation

<sup>2</sup>See its relation to  $\tan^{-1}\left(\frac{y}{x}\right)$  at <https://en.wikipedia.org/wiki/Atan2>.

Before we move on to division, let's see what the multiplicative inverse would look like:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{a + jb} = \frac{1}{a + jb} \times \frac{a - jb}{a - jb} \\ &= \frac{a - jb}{a^2 + b^2} = \frac{a}{a^2 + b^2} - j \frac{b}{a^2 + b^2} \end{aligned} \quad (8)$$

Above, we multiply both the numerator and denominator by  $\bar{z}$ . This allows us to make the denominator real, and we also observe  $\bar{z} \times z = |z|^2$ .

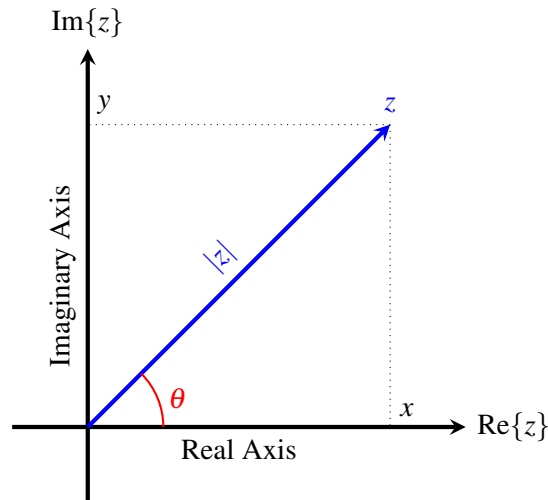
Following the same train of thought, let's define division as well:

$$\frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \quad (9)$$

$$= \frac{(a_1 + jb_1) \times (a_2 - jb_2)}{a_2^2 + b_2^2} \quad (10)$$

$$= \frac{(a_1 * a_2 + b_1 * b_2) - j(a_1 * b_2 - a_2 * b_1)}{a_2^2 + b_2^2} \quad (11)$$

At line (10), we substitute the multiplicative inverse found in (8), and we continue by carrying out the multiplication as defined in (7).



**Figure 1:** Complex number  $z$  depicted as a vector in the complex plane.

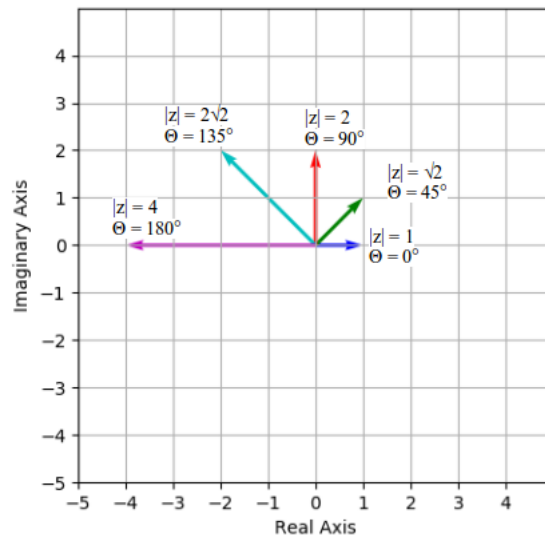
### 1.3 Complex Plane

The complex plane is an extension of the real number line; by adding another independent axis to represent the purely complex numbers (i.e. those with the real part equal to zero) we can represent all complex numbers. As shown in Figure 4, a complex number  $z = x + jy$  has an intercept at  $x$  along the **real axis** and  $y$  along the **imaginary axis**. We can also see visually that the magnitude of  $z$  is its distance from the origin, and the phase is the angle from the positive real axis.

## 2 The Polar Form

### 2.1 Revisiting Multiplication

With our understanding of the complex plane, let's see how a complex number  $z = 1 + j0$  changes as we multiply it with  $z_1 = 1 + j$  repeatedly:



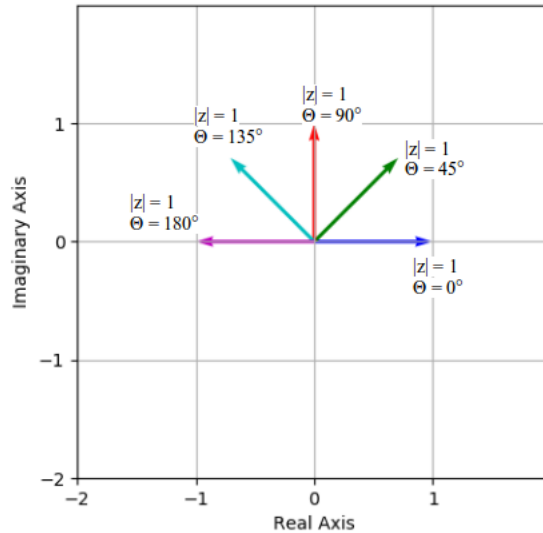
**Figure 2:** Multiplying  $z$  (in blue) by  $z_1$  repeatedly

Clearly, it looks like  $z$  seems to be rotating by  $45^\circ$ . Is this an arbitrary angle? No, the angle of rotation seems to be related to the phase  $\angle z_1 = 45^\circ$ . So, can we think of some representation of complex numbers where this additive property of angles is fulfilled?

How can we turn multiplication into some kind of addition? Well, we can represent the phase in an exponent. What if we used 'e' as the base for our exponent, and just set  $z = e^{\angle z}$ ? Unfortunately, a pure real exponential will blow up as  $\angle z$  increases, but this is not the behaviour we see with complex number angles. It is easy to see that any real number in the exponent couldn't possibly behave the right way — angles spin around while real exponents either go to zero or blow up. So if the thing in the exponent can't be real, for now, let's take a leap of faith and hope that something imaginary works! i.e. Multiply the thing in the exponent by  $j$ , i.e.  $z = e^{j\angle z}$ . We don't yet know how this will behave, but let's push forward.

The above motivation just captured the rotation property of multiplication by complex numbers. Furthermore, there is some scaling as well. More precisely, for the example of  $1 + j$ , it is scaling by  $\sqrt{2}$ . This cannot be a coincidence, so how can we express this scaling with our new form? We could multiply the magnitude of  $z_2$  with the exponential, i.e.  $z_2 = |z_2|e^{j\angle z_2}$ .

Now, let's try to isolate the effect of rotation by multiplying  $z = 1 + j0$  by  $z_2 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$  (note that  $|z_2| = 1$ ) repeatedly. We get the following plot:



**Figure 3:** Multiplying  $z$  (in blue) by  $z_2$  repeatedly

Clearly, we have a rotation by  $45^\circ$  as before, and we do not have any scaling. So, our intuition of multiplying the magnitude with the exponential agrees with this example as well.

## 2.2 Developing the Polar Form and Euler's Equation

In the last section we conjectured,

$$z = |z|e^{j\angle z} \quad (12)$$

But what does  $e^{j\angle z}$  mean? It is a natural first step, from our experiences, to use the exponential's definition in the form of its Taylor's expansion around 0 (or the Maclaurin's expansion of 'e'):

$$z = |z|e^{j\theta} \quad (13)$$

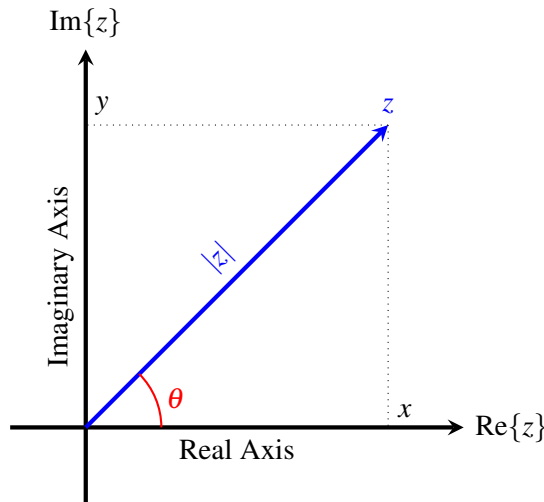
$$= |z| \left( 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \dots + \frac{(j\theta)^{2n}}{2n!} + \frac{(j\theta)^{2n+1}}{(2n+1)!} + \dots \right) \quad (14)$$

$$= |z| \left( 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \dots + (-1)^n \frac{\theta^{2n}}{2n!} + j(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \dots \right) \quad (15)$$

$$= |z| \left[ \left( 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} - \dots + (-1)^n \frac{\theta^{2n}}{2n!} + \dots \right) + j \left( \theta - \frac{\theta^3}{3} + \dots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \dots \right) \right] \quad (16)$$

$$= |z| (\cos(\theta) + j\sin(\theta)) \quad (17)$$

Here,  $\theta = \angle z$  and the angles are clearly measured in radians. Going from equation (16) to (17), we recognize and substitute for the Taylor expansions of sine and cosine around at 0. Let's analyze our result in equation (17) with the complex plane below:



**Figure 4:** Complex number  $z$  depicted as a vector in the complex plane.

Recalling basic definitions from trigonometry, we can see that  $x = |z| \cos(\theta)$  and  $y = |z| \sin(\theta)$ , hence  $z = x + jy = |z| \cos(\theta) + j |z| \sin(\theta)$ , agreeing with our previous result. Hence, the form we guessed in the earlier section was correct and we have come back full circle, connecting with the rectangular form we discussed in Section 1.

**Concept Check:** Using Euler's equation, write sine and cosine as sums of exponentials.

**Solution:**

$$e^{j\theta} = \cos(\theta) + j \sin(\theta) \quad (18)$$

$$e^{-j\theta} = \cos(-\theta) + j \sin(-\theta) = \cos(\theta) - j \sin(\theta) \quad (19)$$

Adding and Subtracting equation (18) and (19), we get :

$$2 \cos(\theta) = e^{j\theta} + e^{-j\theta} \Rightarrow \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad (20)$$

$$2j \sin(\theta) = e^{j\theta} - e^{-j\theta} \Rightarrow \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (21)$$

This is the source of the famous identity  $e^{j\pi} = -1$ , connecting four very fundamental numbers together:  $e, \pi, 1, j$ .

## 2.3 Conclusion

The polar form of  $z$  is a very important representation as it greatly simplifies multiplication. The polar form of  $z = a + jb$  is given as follows:

$$z = |z| e^{j\angle z}$$

The conjugate of  $z$  is given as  $\bar{z} = |z| e^{-j\angle z}$  since  $\sin(-\theta) = -\sin(\theta)$  while  $\cos(-\theta) = \cos(\theta)$  from trigonometry.

Multiplication in this form is given as:

$$z_1 \times z_2 = (|z_1| e^{j\angle z_1}) \times (|z_2| e^{j\angle z_2}) = (|z_1| * |z_2|) e^{j(\angle z_1 + \angle z_2)}$$

If we look carefully, we can realize that complex multiplication is nothing but a rotation operation, followed by scaling. We are essentially rotating  $z_1$  by  $\angle z_2$  in the counter-clockwise direction<sup>3</sup>, and scaling it by a factor of  $|z_2|$ . Of course, this conclusion would have been difficult to make precise without the use of polar forms. We will solidify this view of rotations further in the next section where we will model complex numbers as matrices to augment of the vector intuition we have gained thus far.

---

<sup>3</sup>It is common to define the counter-clockwise direction as the positive direction while measuring angles in the Cartesian plane

## 2.4 Useful Identities

### Complex Number Properties

**Rectangular vs. polar forms:**  $z = x + jy = |z|e^{j\theta}$

where  $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ ,  $\theta = \text{atan2}(y, x)$ . We can also write  $x = |z| \cos \theta$ ,  $y = |z| \sin \theta$ .

**Euler's identity:**  $e^{j\theta} = \cos \theta + j \sin \theta$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \quad \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

**Complex conjugate:**  $\bar{z} = x - jy = |z|e^{-j\theta}$

$$\overline{(z + w)} = \bar{z} + \bar{w}, \quad \overline{(z - w)} = \bar{z} - \bar{w}$$

$$\overline{(zw)} = \bar{z}\bar{w}, \quad \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

$$\bar{\bar{z}} = z \Leftrightarrow z \text{ is real}$$

$\bar{z} = -z \Leftrightarrow z$  is purely complex, i.e. no real part

$$\overline{(z^n)} = (\bar{z})^n$$

### Complex Algebra

Let  $z_1 = x_1 + jy_1 = |z_1|e^{j\theta_1}$ ,  $z_2 = x_2 + jy_2 = |z_2|e^{j\theta_2}$ .

(Note: we adopt the easier representation between rectangular form and polar form.)

**Addition:**  $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

**Multiplication:**  $z_1 z_2 = |z_1| |z_2| e^{j(\theta_1 + \theta_2)}$

**Division:**  $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\theta_1 - \theta_2)}$

**Power:**  $z_1^n = |z_1|^n e^{jn\theta_1}$   
 $z_1^{\frac{1}{2}} = \pm |z_1|^{\frac{1}{2}} e^{j\frac{\theta_1}{2}}$

(Note: Just like square roots are not unique, other fractional powers of  $z_1$  are not unique as well)

### Useful Relations

$$-1 = j^2 = e^{j\pi} = e^{-j\pi}$$

$$j = e^{j\frac{\pi}{2}} = \sqrt{-1}$$

$$-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}}$$

$$\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \frac{\pm(1+j)}{\sqrt{2}}$$

$$\sqrt{-j} = (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{-j\frac{\pi}{4}} = \frac{\pm(1-j)}{\sqrt{2}}$$

**Concept Check:** Verify the above identities for yourself if you have not done so in prior classes.

## 3 Matrices as Complex Numbers

Viewing complex numbers as vector models definitely seems attractive and it does fit into our framework of the complex plane, but it has a major flaw — vectors do not naturally multiply, but complex numbers do. In fact, multiplication is the *raison d'être* for complex numbers. So, how do we get a better model? What both adds and multiplies? Enter matrices, and more specifically rotation matrices.

### 3.1 Matrix form of rotations

But first, what is a rotation matrix? To begin answering this question, we need to first understand what a rotation transformation would look like. Rotating the vector  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  by angle  $\theta$  in the counter clockwise direction would give us  $\vec{\tilde{e}}_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ , and similarly for  $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , we will get  $\vec{\tilde{e}}_2 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$ . Hence,

we can describe the rotation transform (by angle  $\theta$ ) as the following matrix:

$$\tilde{\mathbf{v}} = R_{\theta} \mathbf{v} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{v} \quad (22)$$

An important thing to note is this rotation matrix has orthonormal columns<sup>4</sup>. Next, what would happen if we rotated a vector by  $\theta_1$  and then by  $\theta_2$ ? Well, it would be equivalent to rotating it by  $\theta_1 + \theta_2$ , hence we have:

$$R_{\theta_1} * R_{\theta_2} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} = R_{\theta_1 + \theta_2} \quad (23)$$

**Concept Check:** Use basic trigonometry (in particular, the sum-angle formulas for sine and cosine that you probably derived in high school) to check the equality established in equation (23). Furthermore, rotations in 2D are commutative<sup>5</sup> as well. Show that this is true by proving  $R_{\theta_1} * R_{\theta_2} = R_{\theta_2} * R_{\theta_1}$

When we look back at rotation matrix in (22), it bears some resemblance to the Euler form (equation 17) we discovered in the previous section. If we have a complex number  $z = a + jb = \cos(\theta) + j \sin(\theta)$ , where  $|z| = 1$  (for simplicity, we will look at scaling a bit later) and  $\angle z = \theta$ , then we could define a matrix  $Z_{(a,b)}$  as follows:

$$Z_{(a,b)} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad (24)$$

**Concept Check:** Check that this matrix has orthogonal columns.

What is a complex conjugate in this representation? What can we do to swap the  $b$  and  $-b$  in the matrix above? Indeed we see that transposing the matrix corresponds to complex conjugation of the underlying complex number. It has no effect on a scaled identity matrix which would correspond to a purely real number.

Next, let's look at the scaling. In this case, we have  $z = a + jb$ , with  $|z| = \sqrt{a^2 + b^2}$ . To account for this in our matrix model, we can factor out  $|z|$  as follows:

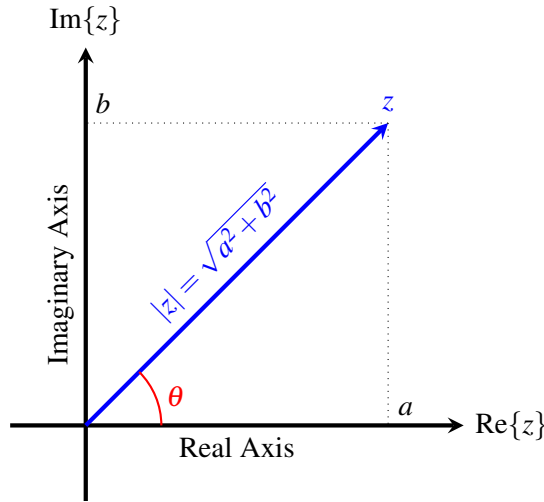
$$Z_{a,b} = \sqrt{a^2 + b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2 + b^2}} & -\frac{b}{\sqrt{a^2 + b^2}} \\ \frac{b}{\sqrt{a^2 + b^2}} & \frac{a}{\sqrt{a^2 + b^2}} \end{bmatrix} \quad (25)$$

Looking at the above form, the factor out in the front is responsible for the scaling. From the figure below, we can find  $\theta = \text{atan2}(b, a)$  such that  $\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}}$  and  $\sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}$ .

<sup>4</sup>The columns in a matrix with orthonormal columns all have norm 1 and are mutually orthogonal to each other. Such matrices are commonly referred to as **orthogonal** matrices in the mathematical literature.

<sup>5</sup>This commutative property for rotations only holds for 2D spaces, and not for 3D spaces. Take a second to think about this!





**Figure 5:** Complex number  $z = a + jb$  represented as a vector in the complex plane.

Now, let's see if this model fits with everything that we know about complex arithmetic.

### 3.1.1 Addition:

For two complex numbers,  $z_1 = a_1 + jb_1$  and  $z_2 = a_2 + jb_2$ , we have:

$$Z_{(a_1, b_1)} + Z_{(a_2, b_2)} = \begin{bmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{bmatrix} = Z_{(a_1 + a_2, b_1 + b_2)}$$

Hence, it satisfies our definition of addition.

### 3.1.2 Multiplication by real number:

Let  $z = a + jb$ , then  $\lambda z = \lambda a + j\lambda b$ , where  $\lambda$  is a real number. This can be easily extended to our matrix form as well:

$$\lambda Z_{(a, b)} = \lambda \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \lambda a & -\lambda b \\ \lambda b & \lambda a \end{bmatrix} = Z_{(\lambda a, \lambda b)}$$

### 3.1.3 Multiplication by a complex number:

Finally, and the reason we are pursuing this representation, multiplication by another complex number. Let  $z_1 = a_1 + jb_1$  and  $z_2 = a_2 + jb_2$ , then we have  $z_1 \times z_2 = (a_1 * a_2 - b_1 * b_2) + j(a_1 * b_2 + a_2 * b_1)$ . Let's check if this is the case with matrix multiplication:

$$Z_{(a_1, b_1)} * Z_{(a_2, b_2)} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 * a_2 - b_1 * b_2 & -(a_1 * b_2 + a_2 * b_1) \\ a_1 * b_2 + a_2 * b_1 & a_1 * a_2 - b_1 * b_2 \end{bmatrix} = Z_{(a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)}$$

Note that since our 2D rotations are commutative, so are the multiplications with complex numbers.

Multiplication by the complex conjugate also clearly gives an identity matrix with the magnitude squared along the diagonal.

It turns out, although we will not show this here, that even the natural generalization of exponentiation to matrices works with this model of complex numbers. We get  $e^{a+jb} = e^a e^{jb} = e^a(\cos b + j \sin b)$ . Actually, the understanding of the natural generalization of exponentiation to matrices requires understanding the solutions to systems of differential equations, where the complex exponentiation case turns out to represent the behavior of RLC circuits. To understand this requires understanding the eigenvalues of the kinds of matrices we find here, but that is a subject of a different note.

**Contributors:**

- Aditya Arun.
- Anant Sahai.
- Nikhil Shinde.