1 RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor over time in an RC circuit. We set up our problem by first defining three functions over time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, and $V_o(t)$ is the voltage across the capacitor at time t.

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_o = \frac{Q}{C}$ where Q is the charge across the capacitor.

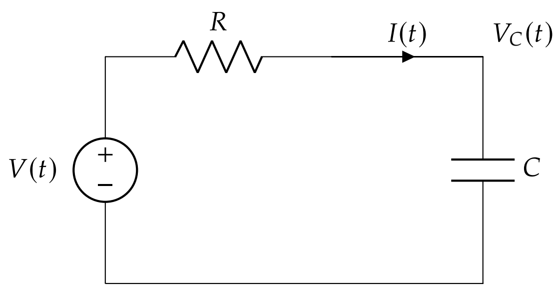

Figure 1: Example Circuit

1. First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.

   Answer

   Differentiating $V_C(t) = \frac{Q(t)}{C}$ in terms of t, we get

   $$\frac{dV_C(t)}{dt} = \frac{dQ(t)}{dt}\frac{1}{C}$$

   By definition, the change in charge is the current across the capacitor, so

   $$\frac{d}{dt}V_C(t) = I(t)\frac{1}{C}$$

2. Write a system of equations that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.

   Answer

   Kirchhoff's law states that the voltage across a closed loop is 0.

   $$RI(t) + V_C(t) - V(t) = 0$$

   $$RI(t) + V_C(t) = V(t) \tag{1}$$

3. Rewrite the previous equation in part (b) in the form of a differential equation involving only $V_C(t)$ and $V(t)$.

   Answer

   From part (a), we have

$$I(t) = \frac{dV_C(t)}{dt}C$$

   Substituting this into Equation 1 gives us

$$RC\frac{dV_C(t)}{dt} + V_C(t) = V(t)$$

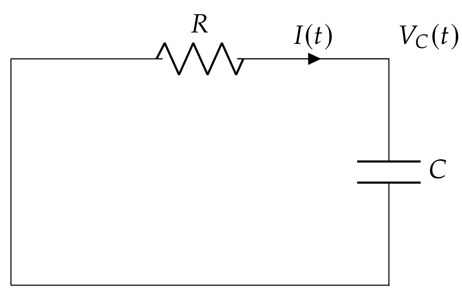

Figure 2: Circuit for part (d)

4. Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

   Answer

   Because $V(t) = 0$, our differential equation simplifies to

$$RC\frac{dV_C(t)}{dt} + V_C(t) = 0$$

   Doing some algebraic manipulations gives us

$$\frac{dV_C(t)}{dt} = -\frac{1}{RC}V_C(t)$$

   This equation tells us that we are looking for some function $V_C(t)$ such that when we take its derivative, we get the same function $V_C(t)$ multiplied by a scalar $-\frac{1}{RC}$. Because the derivative is equal to a scalar times itself, we think that the solution $V_C(t)$ will probably be of the form Ae^{bt}, where A and b are both constants. In this case we see that $b = -\frac{1}{RC}$, and we find that

$$V_C(t) = Ae^{-\frac{1}{RC}t}$$

We still need to solve for the constant A in front of the exponential, and we use $V_C(0) = K$ to help us find A. Setting $t = 0$ in the equation gives us

$$\begin{aligned} V_C(0) &= Ae^{-\frac{1}{RC}0} \\ &= Ae^0 \\ &= A \\ &= V_{DD} \end{aligned}$$

Thus, we see that $A = V_{DD}$, and our solution is

$$V_C(t) = V_{DD}e^{-\frac{1}{RC}t}$$

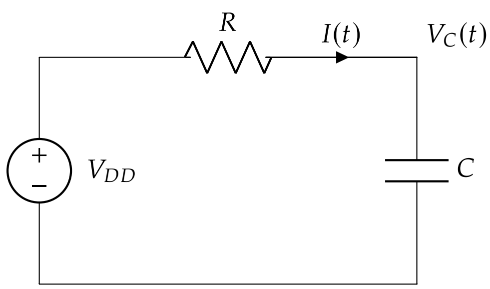

Figure 3: Circuit for part (e)

5. Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer

Substituting $V(t) = V_{DD}$ into our solution from part (c):

$$RC\frac{dV_C(t)}{dt} + V_o(t) = V_{DD}$$

We want to arrange this equation to be in a form that we know how to solve:

$$\frac{d}{dt}V_C = \frac{V_{DD} - V_C(t)}{RC}$$

This is not quite the form we have seen before, as the term on the right is not equal to the term being differentiated. Let's instead define a new variable $\tilde{V}_C(t) = V_C(t) - V_{DD}$. Note that $\frac{d\tilde{V}_C(t)}{dt} = \frac{dV_C(t)}{dt}$. We can substitute these into our differential equation and obtain

$$RC\frac{dV_C(t)}{dt} + V_C(t) - V_{DD} = 0$$

$$RC\frac{d\tilde{V}_C(t)}{dt} + \tilde{V}_C(t) = 0$$

In this equation, we have now removed V_{DD} from the left hand because of how we defined $\tilde{V}_C(t)$. We can now solve the differential equation using the same method as in the previous part to get

$$\tilde{V}_C(t) = Ae^{-\frac{t}{RC}}$$

Substituting $V_C(t) = V_{DD} + \tilde{V}_C(t)$ back into this equation gives us

$$V_C(t) = V_{DD} + Ae^{-\frac{t}{RC}}$$

Using in the initial condition $V_C(0) = 0$, we get:

$$0 = V_{DD} + Ae^{-\frac{0}{RC}} = V_{DD} + A \implies A = -V_{DD}$$

Therefore,

$$\begin{aligned} V_C(t) &= V_{DD} - V_{DD}e^{-\frac{t}{RC}} \\ &= V_{DD}(1 - e^{-\frac{t}{RC}}) \end{aligned}$$

2 Systems of Differential Equations

Consider the following circuit.

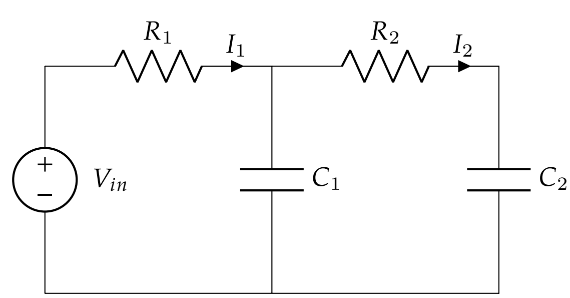

Figure 4: Two dimensional system: a circuit with two capacitors.

1. Write a system of differential equations that governs the voltages V_{C1}, V_{C2} across the capacitors. Use the following values: $C_1 = 1\mu F, C_2 = \frac{1}{3}\mu F, R_2 = \frac{1}{2}M\Omega,\ R_1 = \frac{1}{3}M\Omega$.

   Answer

   Start by solving for the currents and voltages across the capacitors:

$$V_{C2} = V_{C1} - I_{C2}R_2, \quad I_{C2} = C_2\frac{d}{dt}V_{C2}$$

$$I_{C1} = I_{C2} + C_1\frac{d}{dt}V_{C1}, \quad V_{in} - I_{C1}R_1 = V_1$$

   Yields,

$$I_{C1} = \frac{V_{in}}{R_1} - \frac{V_{C1}}{R_1}, \quad I_{C2} = \frac{V_{C1}}{R_2} - \frac{V_{C2}}{R_2}$$

   Now, we can plug into the formula for current across a capacitor:

$$\frac{d}{dt}V_{C1} = \frac{1}{C_1}(I_1 - I_2) = \frac{1}{C_1}(\frac{V_{in}}{R_1} - \frac{V_{C1}}{R_1} - \frac{V_{C1}}{R_1} + \frac{V_{C2}}{R_2}) = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_{C1} + \frac{V_{C2}}{R_2C_1} + \frac{V_{in}}{R_1C_1}$$

$$\frac{d}{dt}V_{C2} = \frac{1}{C_2}(I_2) = \frac{V_{C1}}{R_2C_2} - \frac{V_{C2}}{R_2C_2}$$

   Now group the terms into a matrix with the values given above,

$$\begin{bmatrix} \frac{d}{dt}V_{C1}(t)) \\ \frac{d}{dt}V_{C2}(t) \end{bmatrix} = \begin{bmatrix} -(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_{C1}(t) \\ V_{C2}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{in}(t)$$

   Plugging in the values for R, C yields:

$$\begin{bmatrix} \frac{d}{dt}V_{C1}(t)) \\ \frac{d}{dt}V_{C2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} V_{C1}(t) \\ V_{C2}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{in}(t)$$

2. Suppose also that V_{in} was at 7V for a long time, and then transitioned to be 0V at time $t = 0$. Write the system of differential equations that are valid for $t \geq 0$ in matrix form. What are the initial conditions?

   Answer

$$\begin{bmatrix} \frac{d}{dt}V_{C1}(t) \\ \frac{d}{dt}V_{C2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} V_{C1}(t) \\ V_{C2}(t) \end{bmatrix}$$

$$\begin{bmatrix} V_{C1}(0) \\ V_{C2}(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

   We will define the differential matrix as A_y, where

$$A_y = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$$

3. Find the eigenvalues λ_1, λ_2 and eigenspaces for the matrix corresponding to the differential equation matrix above.

Answer

Eigenvalues λ and eigenvectors v of matrix A are given by

$$A_y v = \lambda v.$$

In order to find the eigenvalues, we take the determinant:

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix}\right) = 0$$

We can solve this using a 2×2 determinant form seen in 16A,

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc,$$

or by Gaussian elimination.

$$\begin{aligned} (-5-\lambda)(-6-\lambda) - 12 &= 0 \\ 30 + 11\lambda + \lambda^2 - 12 &= 0 \\ \lambda^2 + 11\lambda + 18 &= 0 \\ (\lambda+9)(\lambda+2) &= 0 \end{aligned}$$

Giving:

$$\lambda = -9, -2$$

The eigenspace associated with $\lambda_1 = -9$ is given by:

$$\begin{bmatrix} -5+9 & 2 \\ 6 & -6+9 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_{\lambda_1} = \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The eigenspace associated with $\lambda_2 = -2$ is given by:

$$\begin{bmatrix} -5+2 & 2 \\ 6 & -6+2 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_{\lambda_2} = \beta \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$