1 State Space Models

There are many kinds of *dynamical systems* we might want to study or control. Some examples are an airplane's flight, the air inside a building, or network traffic on the internet. We can develop controllers for these systems to regulate particular quantities that we care about, like an autopilot to level an airplane's flight, a thermostat to keep a building at a comfortable temperature, or internet congestion control to manage data rates. Other dynamical systems and controllers can be found in nature, like the biochemical systems that regulate conditions inside a living cell.

When we want to study or control a dynamical system, our first step is usually to write out equations that describe its physics. These equations are called a *model*, and they predict what a system will do over time. We will study systems that change continuously in time like electrical circuits, and systems that evolve in discrete time steps, like the yearly number of professors in EECS.

State variables are a set of variables that fully represent the state of a dynamical system at a given time, like capacitor voltages and inductor currents in electrical circuits. In a mechanical system, they could be the positions and velocities of masses. The state variables can be written together in a *state vector* $\vec{x}(t) \in \mathbb{R}^n$ where n is the number of state variables that describe the system.

2 Continuous Systems

For a continuous system, the dynamics can be described by n first-order differential equations:

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t))$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function of the state vector that returns the time derivative of the state vector (which is an n-dimensional vector containing the time derivative of each state variable).

A system with m input signals can be described as:

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$$

where $\vec{u}(t) \in \mathbb{R}^m$ is a *control input* with which we can vary to influence the system.

We can expand out this vector dynamics equation:

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \\ \frac{d}{dt}x_3(t) \\ \vdots \\ \frac{d}{dt}x_n(t) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}(t), \vec{u}(t)) \\ f_2(\vec{x}(t), \vec{u}(t)) \\ f_3(\vec{x}(t), \vec{u}(t)) \\ \vdots \\ f_n(\vec{x}(t), \vec{u}(t)) \end{bmatrix},$$

where $f_i(\vec{x}, \vec{u}(t))$ returns the time derivative of the ith state variable. This form of the equations works for linear systems and complicated non-linear systems.

For a linear time-invariant system, we can make some simplifications since $f(\vec{x}(t), \vec{u}(t))$ will be a linear combination of the state variables and inputs. Writing it out in the expanded way, this looks like:

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \\ \frac{d}{dt}x_3(t) \\ \vdots \\ \frac{d}{dt}x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n & + & b_{11}u_1 + b_{12}u_2 + \ldots + b_{1m}u_m \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n & + & b_{21}u_1 + b_{22}u_2 + \ldots + b_{2m}u_m \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n & + & b_{31}u_1 + b_{32}u_2 + \ldots + b_{3m}u_m \\ & \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n & + & b_{n1}u_1 + b_{n2}u_2 + \ldots + b_{nm}u_m \end{bmatrix}$$

This is equivalent to a matrix equation of the form $\frac{d}{dt}\vec{x}(t) = \mathbf{A}\vec{x}(t) + \mathbf{B}\vec{u}(t)$:

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \\ \frac{d}{dt}x_3(t) \\ \vdots \\ \frac{d}{dt}x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ b_{31} & b_{32} & \ldots & b_{3m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \ldots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

3 Discrete Time Systems

For a discrete-time system, the dynamics can be described by n difference equations:

$$\vec{x}[t+1] = f(\vec{x}[t], \vec{u}[t]),$$

where $\vec{x}(t+1)$ is the new state vector at the next time step.

As in the continuous case, a linear-time invariant system's dynamics can be written:

$$\vec{x}[t+1] = \mathbf{A}\vec{x}[t] + \mathbf{B}\vec{u}[t]$$

4 Dirty Dishes

I am a trip planner who lodges travellers at Bob's Bed and Breakfast. At the beginning of each day, Bob will do half of the dirty dishes in the sink. During the day, each of his guests will use 4 pounds of dishes minus an eighth pound of dishes for each pound of dishes already in the sink at the beginning of the day (as Bob's kitchen gets too messy).

(a) What is the state vector for Bob's kitchen sink system? What are the inputs? Write out the state space model.

Answer

The dishes in the sink are the state variable x. The number of guests are the input u.

$$x[t+1] = \frac{1}{2}x[t] + \left(4 - \frac{1}{8}x[t]\right)u[t]$$

(b) Is Bob's kitchen sink a linear time-invariant system? If it is, write it in the form $\vec{x}[t+1] = \mathbf{A}\vec{x}[t] + \mathbf{B}\vec{u}[t]$. If it isn't, explain why not.

Answer

No, the state variable is multiplied by the input.

(c) On Wednesday morning (before Bob gets up), there are 4 pounds of dishes in the sink. On Wednesday, Bob has 4 guests, and on Thursday, he has 5 guests. How many pounds of dishes are in the sink after Thursday?

Answer

$$x[1] = \frac{1}{2}(4) + \left(4 - \frac{1}{8}(4)\right)(4) = 16$$
$$x[2] = \frac{1}{2}(16) + \left(4 - \frac{1}{8}(16)\right)(5) = 18$$

(d) I am a very eccentric trip planner and I want Bob to have exactly 12 pounds of dishes in his sink. He has 24 pounds of dishes in his sink. How many guests should I lodge at Bob's Bed and Breakfast today? How many guests should I lodge tomorrow?

Answer

$$12 = \frac{1}{2}(24) + \left(4 - \frac{1}{8}(24)\right)u[0]$$
$$u[0] = 0$$

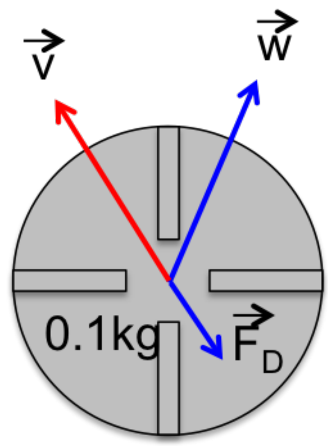
$$12 = \frac{1}{2}(12) + \left(4 - \frac{1}{8}(12)\right)u[1]$$
$$u[1] = \frac{12}{5}$$

5 Remote control tank

I have a toy hovercraft that I can drive around on the ground. It weighs $0.1\,\mathrm{kg}$. My remote control has two levers: one sets the thrust in the x-direction, w_x, measured in Newtons, and the other sets the thrust in the y-direction, w_y, measured in Newtons. The hovercraft experiences a drag force:

$$\vec{F} = -D\vec{v},$$

where $\vec{F}$ is the drag force vector in Newtons, $\vec{v}$ is the hovercraft velocity vector in $\frac{\mathrm{m}}{\mathrm{s}}$, and D is the coefficient $0.05\,\frac{\mathrm{N\,s}}{\mathrm{m}}$.



(a) What are the state variables for the hovercraft? What are the inputs?

Answer

The state vector includes the x and y position and the x and y velocity. It could be arranged as:

$$\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix}$$

The inputs are w_x and w_y. The input vector could be arranged as:

$$\begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

(b) Write out the state space model using the state variables and inputs you identified.

Answer

First, let's write out the force balances in the x and y directions using Newton's second law: $F = ma$.

$$\begin{aligned} ma_x &= w_x - Dv_x \\ a_x &= \frac{1}{m}w_x - \frac{D}{m}v_x \\ ma_y &= w_y - Dv_y \\ a_y &= \frac{1}{m}w_y - \frac{D}{m}v_y \end{aligned}$$

Since $\vec{a} = \frac{d}{dt}\vec{v}$, we can now write out our equations:

$$\begin{aligned} \frac{d}{dt}x &= v_x \\ \frac{d}{dt}y &= v_y \\ \frac{d}{dt}v_x &= \frac{1}{m}w_x - \frac{D}{m}v_x \\ \frac{d}{dt}v_y &= \frac{1}{m}w_y - \frac{D}{m}v_y \end{aligned}$$

Substituting in for D and m, we get:

$$\begin{aligned} \frac{d}{dt}x &= v_x \\ \frac{d}{dt}y &= v_y \\ \frac{d}{dt}v_x &= 10w_x - 0.5v_x \\ \frac{d}{dt}v_y &= 10w_y - 0.5v_y \end{aligned}$$

(c) Is this system linear? If it is, write it in the form $\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$. If it isn't, explain why not.

Answer

This system is linear. Using the state vector from the first part, we can write the dynamics as $\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$ where:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 10 & 0 \\ 0 & 10 \end{bmatrix}$$

(d) I place my hovercraft at $(1, 0)$. At $t = 0$, I gently kick my hovercraft, so that it is moving at $2\,\frac{\mathrm{m}}{\mathrm{s}}$ in the x direction, and I don't touch the remote control. What does the hovercraft do? Where will it be at $t = 10$?

Answer

The inputs are all zero and the y-direction dynamics won't do anthing interesting since $y = 0$ and $v_y = 0$. Let's look at the x dynamics: $\frac{d}{dt}v_x = -0.5v_x$ and $\frac{d}{dt}x = v_x$. v_x's dynamics look like a homogeneous first order differential equation, and we can find x by integrating v_x. From our knowledge of first order differential equations,

$$v_x(t) = 2e^{-0.5t}$$

Integrating v_x and plugging in our initial conditions, we get

$$x(t) = 5 - 4e^{-0.5t}$$

This means that at $t = 10$, x is at $5 - 4e^{-5}$, which is very close to $5\,\mathrm{m}$.