1 Jacobian Warm-Up

Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_1 x_2^2 \\ x_1 \end{bmatrix}$$

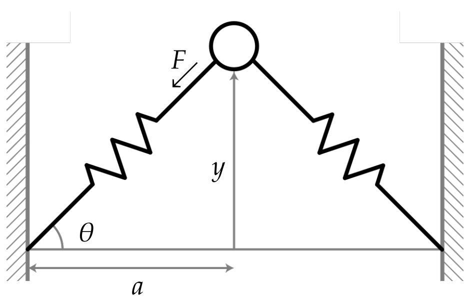
Calculate its Jacobian.

Answer

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}\vec{x}} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_1 + x_2^2 & 2x_1 x_2 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

2 Linearization

Consider a mass attached to two springs:



We assume that each spring is linear with spring constant k and resting length X_0. We want to build a state space model that describes how the displacement y of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt^2} = -\frac{2k}{m}(y - X_0 \frac{y}{\sqrt{y^2+a^2}})$.

a) Write this model in state space form $\dot{x} = f(x)$.

Answer

We introduce states $x_1 = y$ and $x_2 = \dot{y}$. Writing the model in state space form gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-2k}{m}\left(x_1 - X_0 \frac{x_1}{\sqrt{x_1^2 + a^2}}\right) \end{bmatrix}.$$

b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

Answer

We find the equilibrium by solving $0 = \dot{x} = f(x)$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-2k}{m}\left(x_1 - X_0 \frac{x_1}{\sqrt{x_1^2 + a^2}}\right) \end{bmatrix}.$$

The unique solution is the equilibrium at $(x_1, x_2) = (0, 0)$.

c) Linearize your model about the equilibrium.

Answer

$$\left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=(0,0)} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - X_0 \frac{a^2}{(x_1^2 + a^2)^{3/2}}\right) & 0 \end{bmatrix}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & 0 \end{bmatrix}$$

So the linearized system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & 0 \end{bmatrix} x.$$

3 Discretization

Consider a cart of mass M, pushed with a force $u(t)$ with position, $p(t)$, and velocity, $v(t)$. Hence, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = v(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = \frac{u(t)}{M}$$

We will apply a constant input between any time $t \in [t, t+T)$. Here T is our time step.

Find a discretized system of equations for this system.

Answer

To discretize this we calculate the change in position and velocity in time T.

$$v(t+T) - v(t) = \int_t^{t+T} \frac{u(\tau)}{M}\,\mathrm{d}\tau = T\frac{u(t)}{M}$$
$$p(t+T) - p(t) = \int_t^{t+T} v(\tau)\,\mathrm{d}\tau$$

Substitute $v(\tau) = v(t) + \int_t^{\tau} \frac{u(\zeta)}{M}\,\mathrm{d}\zeta = v(t) + (\tau - t)\frac{u(t)}{M}$. (the "point-slope formula")

$$= \int_t^{t+T} \left[v(t) + (\tau - t)\frac{u(t)}{M}\right] \mathrm{d}\tau$$
$$= \int_t^{t+T} v(t)\,\mathrm{d}\tau + \int_t^{t+T} (\tau - t)\frac{u(t)}{M}\,\mathrm{d}\tau$$
$$= Tv(t) + \int_t^{t+T} (\tau - t)\frac{u(t)}{M}\,\mathrm{d}\tau$$

Change the variable of integration to $\zeta = \tau - t$:

$$= Tv(t) + \frac{u(t)}{M}\int_0^{T} \zeta\,\mathrm{d}\zeta$$
$$= Tv(t) + \frac{T^2}{2}\frac{u(t)}{M}$$

Therefore we have,

$$p(t+T) = p(t) + Tv(t) + \frac{T^2}{2}\frac{u(t)}{M}$$
$$v(t+T) = v(t) + T\frac{u(t)}{M}$$
$$\begin{bmatrix} p(t+T) \\ v(t+T) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2M} \\ \frac{T}{M} \end{bmatrix} u(t)$$