1 Controllability

We are given a discrete time state space system, where $\vec{x}$ is our state vector, A is the state space model, B is the input matrix, and $\vec{u}$ is the control input.

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

We want to know if this system is "controllable"; if given set of inputs, we can get the system from any initial state to any final state. This has an important physical meaning; if a physical system is controllable, that means that we can get anywhere in the state space. If a robot is controllable, it is able to travel anywhere in the system it is living in (given enough control inputs).

Construting the Controllability Matrix

To figure out if a system is controllable, we can simplify the problem. If we want to reach any final state from any initial state, we can consider the initial state as the origin and the final state as any arbitrary point in the state space. A system is controllable if we start off at the initial state $\vec{x}[0] = \vec{0}$ at time $t = 0$, and after some set of control inputs $\vec{u}[t]$, we can reach an arbitrary final state $\vec{x}_0$. Let's start the system off at $\vec{x}[0]$ and see how the system evolves with each time step.

$$\vec{x}[1] = A\vec{x}[0] + B\vec{u}[0] = A\vec{0} + B\vec{u}[0] = B\vec{u}[0]$$

This shows us that we can go anywhere spanned by B in the first time step. Using our input vector $\vec{u}$, we can push the system anywhere the matrix B lets us go. Now consider the next time step.

$$\begin{aligned}\vec{x}[2] &= A\vec{x}[1] + B\vec{u}[1] \\ &= AB\vec{u}[0] + B\vec{u}[1]\end{aligned}$$

Similarly, at this time step, we can go anywhere spanned by $\begin{bmatrix} B & AB \end{bmatrix}$. Every time step adds another degree of freedom to the system.

If we go another time step, $\vec{x}[3]$, we get the following:

$$\begin{aligned}\vec{x}[3] &= A\vec{x}[2] + B\vec{u}[2] \\ &= A^2B\vec{u}[0] + AB\vec{u}[1] + B\vec{u}[2]\end{aligned}$$

After k time steps, we get the following:

$$\begin{aligned}\vec{x}[k] &= A\vec{x}[k-1] + B\vec{u}[k-1] \\ &= A^{k-1}B\vec{u}[0] + A^{k-2}B\vec{u}[1] + A^{k-3}B\vec{u}[2] + \cdots + AB\vec{u}[k-2] + B\vec{u}[k-1]\end{aligned}$$

After 1 time step, we can go anywhere in the set of vectors spanned by B, after 2 time steps, we can go anywhere spanned by $\begin{bmatrix} B & AB \end{bmatrix}$, and after k time steps, we can go anywhere spanned by the columns of the matrix $\mathcal{C}$ defined below. This is called the "controllability" matrix.

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{k-2}B & A^{k-1}B \end{bmatrix}$$

If this matrix is of rank n (the dimension of our state space), then our system is controllable. It means that our control system is a surjection from the domain of control inputs to the state space. But what if these aren't enough steps and the system can be controlled only in $k+1$ steps? What is the maximal number of steps we need to take to have a long sequence of control inputs that $\{B, AB, A^2B, \ldots\}$ spans the state space?

Cayley-Hamilton Theorem

These questions are answered by the Cayley-Hamilton theorem. The Cayley-Hamilton theorem says that higher order powers of A can be expressed as a linear combination of lower order matrix powers of A. Specifically if A is an $n \times n$, matrix, the highest order unique power of A is A^{n-1}. Thus, if we keep applying control inputs past n time steps, our control inputs will be a linear combination of the previous control inputs and cannot increase the rank of the controllability matrix.

Definition of Controllability

This also works for continuous time systems; instead of incrementing the time steps in our system by 1 every time, we increment by Δt, $2\Delta t$, etc. The math and our controllability test work out to be exactly the same! Putting all of this together, we get the following:

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) \quad \text{or} \quad \vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$
$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-2}B & A^{n-1}B \end{bmatrix}$$

Given a continuous or discrete time system $\vec{x}$ of dimension n, the system is controllable if its controllability matrix $\mathcal{C}$ is of rank n. If a system is controllable, then given a starting position $\vec{x}[0] = \vec{0}$, it takes a maximum of n control inputs over n time steps for the system to reach any final state $\vec{x}_0$.

2 Deadbeat Control

Consider the system

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t].$$

a) Is this system controllable?

Answer

We compute the controllability matrix C:

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

This matrix has a rank of 2, so the system is controllable.

b) For which initial states $\vec{x}[0]$ is there a control that will bring the state to zero in a single time step?

Answer

To find the initial states that can be brought to zero in a single step, we solve:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[0] \\ &= \begin{bmatrix} x_1[0] - x_2[0] \\ x_2[0] - x_1[0] + u[0] \end{bmatrix} \\ \implies 0 &= x_1[0] - x_2[0]. \end{aligned}$$

Therefore, there is a one-dimensional subspace $\{x_1[0] - x_2[0] = 0\}$ of initial states that can be brought to zero in one step.

c) For which initial states $\vec{x}[0]$ is there a control that will bring the state to zero in two time steps?

Answer

To find the initial states that can be brought to zero in two steps, we solve:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[1] \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1[0] - x_2[0] \\ x_2[0] - x_1[0] + u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[1] \\ &= \begin{bmatrix} 2x_1[0] - 2x_2[0] - u[0] \\ 2x_2[0] - 2x_1[0] + u[0] + u[1] \end{bmatrix} \end{aligned}$$

Therefore, any initial state can be brought to zero in two steps using an appropriate choice of inputs $u[0]$ and $u[1]$.

3 Cayley and Hamilton

Cayley is trying to control the system

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t]$$

a) Is this system controllable?

Answer

We compute the controllability matrix C:

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

This matrix has rank 1, so the system is not controllable.

b) Cayley has been trying to find some k, such that the matrix

$$C_k = \begin{bmatrix} B & AB & A^2B & \cdots & A^kB \end{bmatrix}$$

has rank 2 but still hasn't found one. Confirm that for $k = 3$, this matrix still has rank 1.

Answer

$$C_3 = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -2 & 4 & -8 \end{bmatrix}$$

This matrix still has rank 1.

c) Cayley's friend Hamilton remembers hearing somewhere that for any $n \times n$ matrix A, the matrix A^n can always be written as a linear combination of $A^{n-1}, A^{n-2}, \ldots, A$, and I. [1] Is this true for the A matrix of Cayley's system?

Answer

We want to find some coefficients α and β, such that

$$\begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} = A^2 = \alpha A + \beta I = \begin{bmatrix} \beta + \alpha & -3\alpha \\ -3\alpha & \beta + \alpha \end{bmatrix}.$$

If we choose $\alpha = 2$ and $\beta = 8$, we can make this equation hold.

d) Will Cayley ever find some k to make

$$C_k = \begin{bmatrix} B & AB & A^2B & \ldots & A^kB \end{bmatrix}$$

have rank 2?

[1]Hamilton is right about this. It follows from the Cayley-Hamilton Theorem, which says that any $n \times n$ matrix always satisfies its characteristic equation. Therefore, the characteristic equation $\lambda^2 - 2\lambda - 8$ we derived above implies that $A^2 - 2A - 8 = 0$. You'll learn more about this theorem if you take the advanced control course EE 221A.

Answer

No. Since A^2 is just a linear combination of A and I, repeatedly exponentiating A will never get him any more linearly independent matrices. Therefore, for any k, he will never be able to make C_k have rank 2.