Calculating the Singular Value Decomposition

Suppose we have a matrix A of dimension $m \times n$ ($n > m$) with rank r.

We can find the singular value decomposition (SVD)

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

with the following steps.

1. Find the eigenvalues λ_i of $A^T A$ and order them such that $\lambda_1 \geq \ldots \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$.

2. Find the orthonormal eigenvectors of $A^T A$, so that

$$A^T A \vec{v}_i = \lambda_i \vec{v}_i, \qquad i = 1, \ldots, r$$

   Note that the vectors must be orthonormal, that is $\vec{v}_i^T \vec{v}_i = 1$ and $\vec{v}_i^T v_j = 0$ for $i \neq j$.

3. Let $\sigma_i = \sqrt{\lambda_i}$ and set

$$\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}, \qquad i = 1, \ldots, r$$

Note: We will see later that real symmetric matrices $Q = Q^T$ have real eigenvalues and a set of real, orthonormal eigenvectors. Moreover if we can write $Q = R^T R$, the eigenvalues are non-negative.

1 SVD Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

a) Find the SVD of A.

Answer

Using the steps above

1. Compute $A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$ and find its eigenvalues, $\lambda_1 = 18$ and $\lambda_2 = 0$
2. The correspoding eigenvectors are

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

3. Therefore, there is one singular value $\sigma_1 = \sqrt{18} = 3\sqrt{2}$. We find

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

The SVD of A is therefore

$$A = 3\sqrt{2} \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

b) Find the rank of A.

Answer

A has 1 nonzero singular value. So A has rank 1.

c) Find a basis for the nullspace of A.

Answer

We want the basis for the set of vectors $\{\vec{v} \mid A\vec{v} = 0\}$. We've already computed that $A^T A\vec{v}_2 = 0$, so we can check that $\vec{v}_2$ works

$$\text{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$$

d) Find a basis for the range (or columnspace) of A.

Answer

$$\text{range}(A) = \text{span}\left\{\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}\right\}$$

e) Create the SVD of A^T. What are the relationships between the answers to (a)-(d) for A and for A^T?

Answer

We follow the same steps as above, with $B = A^T$

1. We see that

$$B = A^T = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$B^T B = AA^T = \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}$$

$$\lambda = 18, 0, 0$$

2. We find the eigenvector for $\lambda_1 = 18$,

$$\vec{v}_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}.$$

3. We set $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and find

$$\vec{u}_1 = \frac{A^T \vec{v}_1}{\sigma_1} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

Thus, we write

$$B = A^\top = 3\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \end{bmatrix}$$

At this point, we already know the rank is 1. The column space of A^T is

$$\text{range}(A^T) = \text{span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

Observe that the vectors in the column space of A^T together with those in the nullspace of A span $\mathbb{R}^2$.

The two vectors in the nullspace of A^T are

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Similarly, the vectors in the nullspace of A^T together with those in the column space of A span $\mathbb{R}^3$.

Note, if we had just noticed

$$A^T = (\sigma_1 \vec{u}_1 \vec{v}_1^T)^T = \sigma_1 \vec{v}_1 \vec{u}_1^T$$

where $\vec{u}_1, \vec{v}_1$ are as defined for our original A, we could've skippepd many steps for SVD calculation.