Continuous and discrete time

There are two different dialects for modeling change over time. Thus far we have modeled real-life events using differential equations and initial conditions. For example, the voltage across a capacitor connected to a voltage source by a resistor is fully described by the following differential equation and initial conditions.

$$\frac{\mathrm{d}}{\mathrm{d}t} v_C(t) = -\frac{1}{RC} v_C(t) + \frac{1}{RC} v_{\mathrm{in}}(t), \quad v_C(0) = v_0 \tag{1}$$

Abstracting away particulars, *continuous-time* scalar linear systems can be represented in variants of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) + \mu u(t), \quad x(0) = x_0. \tag{2}$$

This discussion will introduce *discrete-time* scalar linear systems, which have models similar to the following:

$$x[t+1] = a x[t] + b u[t], \quad x[0] = x_0. \tag{3}$$

Notice that evolution is represented by defining the transition from $x[t]$ to $x[t+1]$. The state x is not a continuous function of time, but a sequence of individual moments. Can you think of systems in life that are naturally more susceptible to discrete-time modeling?

1 Differential equations with piecewise constant inputs

1. Let $x(\cdot)$ be a solution to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda \left(x(t) - u(t) \right). \tag{4}$$

Let $T > 0$. Let $x[\cdot]$ "sample" $x(\cdot)$ as follows:

$$x[n] = x(nT). \tag{5}$$

Assume that $u(\cdot)$ is constant between samples of $x(\cdot)$, i.e.

$$u(t) = u[n] \quad \text{when} \quad nT \leq t < (n+1)T. \tag{6}$$

For a general time-step n, write $x[n+1]$ in terms of $x[n]$ and $u[n]$. Conclude that the sampled system of a continuous-time linear system is in fact a discrete-time linear system.

2. Let $T = 1$ and $\lambda = -100$. Sketch a piecewise constant input $u[\cdot]$ of your choice, then sketch $x(t)$. Mark $x[n]$. Your sketch doesn't have to be exact, but you should be able to supply analysis to justify why it looks a certain way: how are you using the fact that λT is large and negative?

3. Let $T = 1$ and $\lambda = -1$. Define $u[n]$ as follows:

$$u[n] = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}. \tag{7}$$

Sketch $x(t)$.