1 Changing Coordinates and Systems of Differential Equations, pt. 2

In the previous discussion we analyzed the differential equation and eigenvalues for the voltages V_{C1}, V_{C2} across the capacitors shown in Figure 1. The devices are set to the following values: $C_1 = 1\mu F, C_2 = \frac{1}{3}\mu F, R_2 = \frac{1}{2}M\Omega,\ R_1 = \frac{1}{3}M\Omega$. Remember that in the case where V_{in} starts at $7V$ then transitions to $0V$ at $t = 0$, the differential equation can be expressed as:

$$\frac{d}{dt}y_1(t) = -5y_1(t) + 2y_2(t) \tag{1}$$

$$\frac{d}{dt}y_2(t) = 6y_1(t) - 6y_2(t) \tag{2}$$

with initial conditions $y_1(0) = 7$ and $y_2(0) = 7$. Or equivalently in matrix form as:

$$A_y = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$$

We found the eigenvalues $\lambda_1,\ \lambda_2$ for the matrix to be:

$$\lambda = -9, -2$$

The eigenspace associated with $\lambda_1 = -9$ is given by:

$$\vec{v}_{\lambda_1} = \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The eigenspace associated with $\lambda_2 = -2$ is given by:

$$\vec{v}_{\lambda_2} = \beta \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

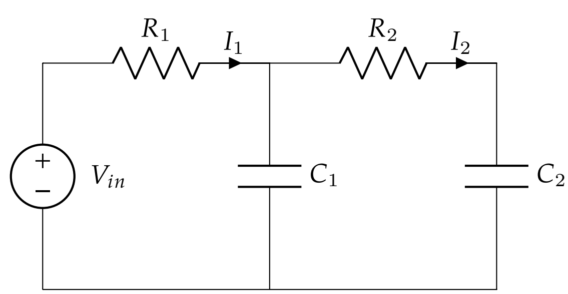

Figure 1: Two dimensional system: a circuit with two capacitors, like the one in lecture.

The reason we spent the time in the previous discussion to find the eigenvalues and eigenspaces is because we can use them as a convenient basis for coordinate transformation! Today we'll solve this differential equation in a simple way with this methodology.

1. Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t)$, $z_{\lambda_2}(t)$. (These variables represent eigenbasis-aligned coordinates.)

2. Solve the differential equation for $z_{\lambda_i}(t)$ in the eigenbasis.

3. Convert your solution back into the original coordinates to find $y_i(t)$.

4. We can solve this equation using a slightly shorter approach by observing that the solutions for $y_i(t)$ will all be of the form

   $$y_i(t) = \sum_k K_{i,k} e^{\lambda_k t}$$

   where λ_k is an eigenvalue of our differential equation relation matrix and the $K_{i,k}$ are constants derived from our initial conditions and the coordinate changes involved.

   Since we have observed that the solutions will include $e^{\lambda_i t}$ terms, once we have found the eigenvalues for our differential equation matrix, we can guess the forms of the $y_i(t)$ as

   $$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} \\ \gamma e^{\lambda_1 t} + \kappa e^{\lambda_2 t} \end{bmatrix}$$

   where $\alpha,\ \beta,\ \gamma,\ \kappa$ are all constants.

   Take the derivative to write out

   $$\begin{bmatrix} \frac{d}{dt} y_1(t) \\ \frac{d}{dt} y_2(t) \end{bmatrix}.$$

   and connect this to the given differential equation.

   Solve for $y_i(t)$ from this form of the derivative.

2 Intro to Complex Numbers

There are some matrices whose eigenvalues are not real numbers. One such matrix is the rotation matrix R that rotates a vector by 90 degrees counter clockwise:

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

1. Write out the equation that the eigenvalues of R must satisfy.
2. Let us take j to be the number such that $j^2 = -1$. Using j, solve the previous equation for λ.