1 Eigenvalues of an upper triangular matrix

We will show, without determinants, that the eigenvalues of an upper triangular matrix are its diagonal entries.

Throughout this problem, $A \in \mathbb{C}^{n \times n}$ will be a general square matrix, and a_{ij} will denote its coordinate at row i, column j.

An eigenvalue of A is a scalar $\lambda \in \mathbb{C}$ such that $A - \lambda I$ does not have full rank. Observe this condition is equivalent to the existence of a nonzero vector $\vec{v} \in \mathbb{C}^n$ such that $A\vec{v} = \lambda\vec{v}$.

a) As A is upper triangular, it has the following form:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & 0 & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} \end{bmatrix}. \tag{1}$$

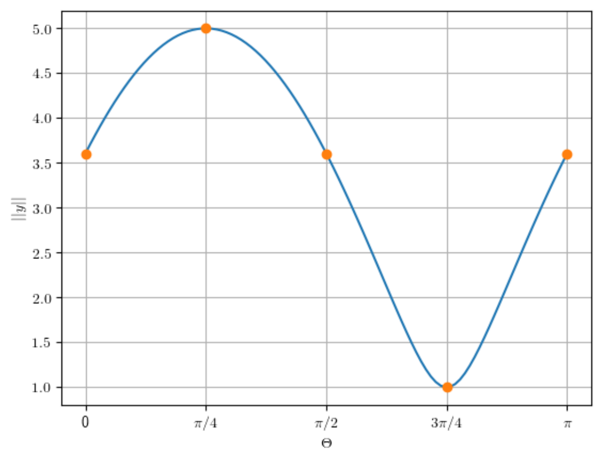
Show that if $a_{k,k}$ is any diagonal value of A, $A - a_{k,k}I$ does not have full rank. (*Hint: how do you check the rank of a matrix by row reduction?*)

b) Show that if $A - \lambda I$ does not have full rank, λ is equal to a diagonal value of A.

2 SVD (40 points)

a) Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $\vec{x} \in \mathbb{R}^n$ be a nonzero vector. Prove that $\left\|A\vec{x}\right\| \geq \sigma_{\min} \left\|\vec{x}\right\|$.

b) Let $A \in \mathbb{R}^{2 \times 2}$ and $\vec{x} = \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$, $||\vec{x}|| = 1$. Now let $\vec{y} = A\vec{x}$. Below is the plot of $||\vec{y}||$ vs θ.



A has the SVD $U\Sigma V^\top$. Either specify what the matrices U, Σ, and V are; or state they they cannot be determined from the information given.

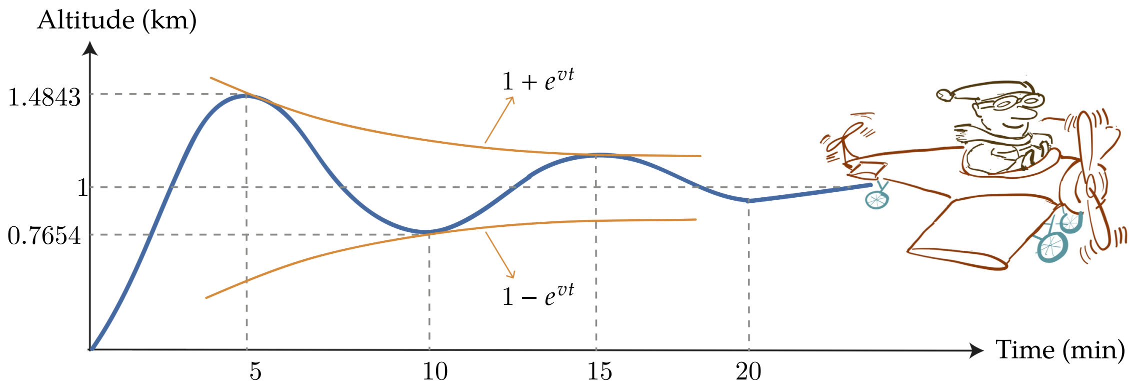
c) Let $A \in \mathbb{R}^{N\times N}, B \in \mathbb{R}^{N\times N}$ be full rank matrices and let $\vec{x} \in \mathbb{R}^N$ have $||\vec{x}|| = 1$. Let $\vec{y} = AB\vec{x}$. Find a upper bound for $||\vec{y}||$ in terms of the singular values of A and B. Explain your answer.

3 Otto the Pilot

Otto has devised a control algorithm, so that his plane climbs to the desired altitude by itself. However, he is having oscillatory transients as shown in the figure. Prof. Arcak told him that if his system has complex eigenvalues

$$\lambda_{1,2} = v \pm j\omega,$$

then his altitude would indeed oscillate with frequency ω about the steady state value, $1\,\mathrm{km}$, and that the time trace of his altitude would be tangent to the curves $1 + e^{vt}$ and $1 - e^{vt}$ near its maxima and minima respectively.



a) Find the real part v and the imaginary part ω from the altitude plot.

b) Let the dynamical model for the altitude be

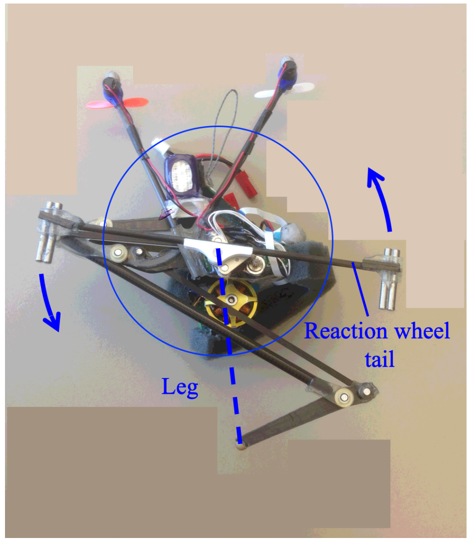
$$\frac{d}{dt}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and a_1 and a_2 are constants. Using your answer to part (a), find what a_1 and a_2 are.

c) Otto can change a_2 by turning a knob. Tell him what value he should pick so that he has a "critically damped" ascent with two real negative eigenvalues at the same location.

4 Balance — linearizing a vector system

Justin is working on a small jumping robot named Salto. Salto can bounce around on the ground, but Justin would like Salto to balance on its toe and stand still. In this problem, we'll work on systems that could help Salto balance on its toe using its reaction wheel tail.



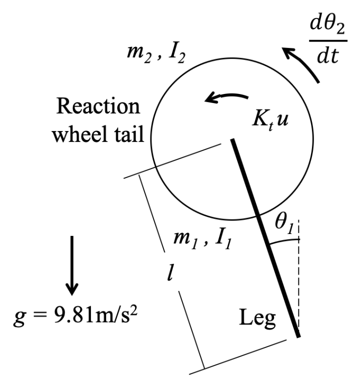


Figure 1: Picture of Salto and the x-z physics model. You can watch a video of Salto here: `https://youtu.be/ZFGxnF9SqDE`

Standing on the ground, Salto's dynamics in the x-z plane (called the sagittal plane in biology) look like an inverted pendulum with a flywheel on the end,

$$\begin{aligned}(I_1 + (m_1 + m_2)l^2)\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} &= -K_t u(t) + (m_1 + m_2)lg\sin\big(\theta_1(t)\big)\\ I_2\frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} &= K_t u(t),\end{aligned}$$

where $\theta_1(t)$ is the angle of the robot's body relative to the ground at time t ($\theta_1 = 0$ rad means the body is exactly vertical), $\frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t}$ is the robot body's angular velocity, $\frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t}$ is the angular velocity of the reaction wheel tail, and $u(t)$ is the current input to the tail motor. $m_1, m_2, I_1, I_2, l, K_t$ are positive constants representing system parameters (masses and angular momentums of the body and tail, leg length, and motor torque constant, respectively) and $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is the acceleration due to gravity.

Numerically substituting Salto's physical parameters, the differential equations become:

$$\begin{aligned}0.001\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} &= -0.025u(t) + 0.1\sin\big(\theta_1(t)\big)\\ 5(10^{-5})\frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} &= 0.025u(t)\end{aligned}$$

a) Using the state vector $\begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix}$, and input u, linearize the system about the point $\vec{x}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ with nominal input $u^* = 0$. **Write the linearized equation as $\frac{d}{dt}\vec{x} = A\vec{x} + Bu$. Write out the matrices with the physical numerical values.**

Note: Since the tail is like a wheel, we care only about the tail's angular velocity $\frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t}$ and not its angle $\theta_2(t)$. This is why $\theta_2(t)$ is not a state.

Hint: The $\sin$ is the only nonlinearity that you have to deal with here.

It is just that single entry in the matrix with the $\sin$ that needs to be approximated for small θ_1, and it is clear that there, $\frac{\sin(\theta_1)}{\theta_1} \approx 1$.

b) Your linearized system should have at least one eigenvalue that corresponds to a growing exponential. If we just do the formal test for controllability by checking the $(A, \vec{b})$ pair for the linearized system, **does it indicate that we could place the closed-loop eigenvalues wherever we want for the linearized system?**

c) Using state feedback, Justin has selected the control gains $K = \begin{bmatrix} 20 & 5 & 0.01 \end{bmatrix}$ for his input $u = K\vec{x}$. **What are the eigenvalues of the closed loop dynamics for the given K?**

Feel free to use numpy.