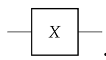
This homework is optional.

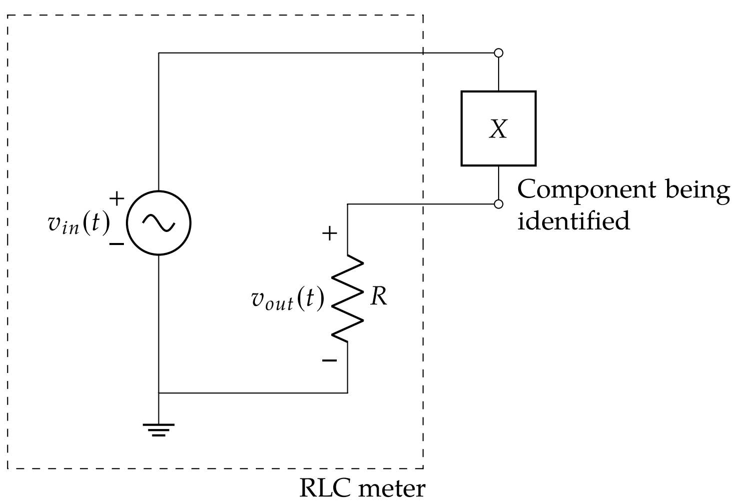
Solutions will be published soon, after you have had some time to try out the problems.

1 Identifying an Unknown Circuit Component

Suppose we have an unknown circuit component, which we'll denote as X and represent with the symbol 

. X could either be a resistor, a capacitor, or an inductor, but we don't know which one it is, nor do we know what its *component value* (that is, its resistance, capacitance, or inductance) could be. If you needed to identify X, that is figure out what kind of component X is and figure out its value, you would use a tool called an *RLC meter*. **In this problem, you will examine how an RLC meter can identify unknown circuit components with the help of transfer functions**.

In circuit form, an RLC meter looks like this:



Here, $v_{in}(t) = A_{in}\cos(2\pi f_0 t + \theta_{in})$ is a known sinusoidal test input of known frequency f_0, known amplitude A_{in}, and known phase θ_{in}; while R is also a known resistance. Under this setup, we know that $v_{out}(t)$ will also be a sinusoid, which we'll denote as $v_{out}(t) = A_{out}\cos(2\pi f_0 t + \theta_{out})$.

When X is connected to the RLC meter, an on-board microcontroller takes samples from $v_{in}(t)$ and $v_{out}(t)$ and uses these samples to compute $Z_X|_{f_0}$, the impedance of the unknown component at frequency f_0. From the value of $Z_X|_{f_0}$, it can figure out whether X is a resistor, a capacitor, or an inductor, as well as what resistance, capacitance, or inductance it has.

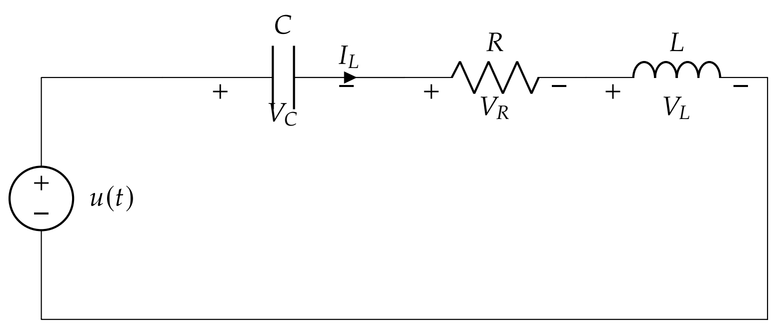
a) **Find the transfer function** $H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$ **when the unknown component is connected to the RLC meter**. Here, $\widetilde{V}_{out}$ and $\widetilde{V}_{in}$ denote the phasor representations of $v_{out}(t)$ and $v_{in}(t)$, respectively. Answer in terms of R, the known resistance, and $Z_X(\omega)$, the unknown impedance.

b) Suppose that we know $H(\omega_0)$, that is the (possibly complex) numerical value of $H(\omega)$ at the angular frequency $\omega_0 = 2\pi f_0$. **Show how to use the value of** $H(\omega_0)$ **to calculate** $Z_X|_{f_0}$**, the impedance of the unknown component at the frequency** f_0. Your result should be an equation for $Z_X|_{f_0}$ in terms of quantities whose values we know.

c) Suppose that we knew $Z_X|_{f_0}$. **Describe how to use** $Z_X|_{f_0}$ **to determine both what kind of component** X **is and the corresponding component value?** *(HINT: Physical resistances, capacitances, and inductances are always positive. And* $\frac{1}{j} = -j$ *for* $j = \sqrt{-1}$*.)*

2 Circuit Discretization

Let's consider the following RLC circuit that you have encountered before.



a) Find the matrix differential equation for the above system using the state-vector $\vec{x} = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$ as

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t).$$

What is A**? What is** $\vec{b}$**?**

Your answers should be in terms of R, L, C.

b) Now, assume for some specific component values we get the following differential equation:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t). \tag{1}$$

Unfortunately, we are unable to measure our state vector continuously. Suppose that we sample the system with some sampling interval T. Let us discretize the above system. Assume that we use piecewise constant voltage inputs $u(t) = u_d(k)$ for $t \in [kT, (k+1)T)$.

Recall from the homework that for a hypothetical scalar differential equation $\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$, we can discretize it as long as $\lambda \neq 0$ as follows:

$$x_d(k+1) = e^{\lambda T} x_d(k) + \frac{e^{\lambda T} - 1}{\lambda} b u_d(k). \tag{2}$$

Here $x_d(k) = x(kT)$.

Using equation (2), calculate the discrete-time system for Equation (1)'s continuous-time vector system in the form:

$$\vec{x}_d(k+1) = A_d \vec{x}_d(k) + \vec{b}_d u_d(k).$$

More concretely, find A_d and $\vec{b}_d$.

You do not need to multiply out any matrices. It is fine if you give your answers as explicit products of matrices/vectors/etc.

Hint: We have provided information regarding the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ in (1) for your convenience (not all of this is needed) on the opposite page.

a) The determinant of A: $\det(A) = 2$.

b) The trace of A: $\mathrm{tr}(A) = -3$.

c) $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$.

d) We can diagonalize the matrix as $A = V\Lambda V^{-1}$, where, Λ is a diagonal matrix with the eigenvalues in its diagonal and the columns of V are the eigenvectors of the corresponding eigenvalues

e) The eigenvalues/eigenvectors for A are:

$$\text{For } \lambda_1 = -2 : \ \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{For } \lambda_2 = -1 : \ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

f) For $V = [\vec{v}_1, \vec{v}_2]$, we have $V^{-1} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

3 SVD stuff

a) Compute the SVD of the following matrix. Express your answer in the the form of $\sum_i \sigma_i \vec{u}_i \vec{v}_i^\top$

$$A = \begin{bmatrix} \vec{a} & -\vec{a} \end{bmatrix}$$

Here, $\vec{a}$ is some arbitrary vector in $\mathbb{R}^n$

b) Compute the compact form SVD of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

4 Control Question

Given a non-linear two-dimensional system with states x_0 and x_1 and inputs u_0 and u_1 that evolves according to the following coupled differential equations:

$$\begin{aligned} \frac{d}{dt}x_0 &= \dot{x}_0 = x_1 \\ \frac{d}{dt}x_1 &= \dot{x}_1 = \alpha - \beta\frac{u_1^2}{x_0^2} \end{aligned} \tag{3}$$

where: $\alpha, \beta > 0$ and $u_1 \geq 0$

a) Write the non-linear system (3) in a vector form $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t))$

b) Find an input vector $\vec{u}_e$ and a state vector $\vec{x}_e$ so that the system remains in the state vector $\vec{x}_e = \begin{bmatrix} x_{0_e} \\ x_{1_e} \end{bmatrix} = \begin{bmatrix} 1 \\ x_{1_e} \end{bmatrix}$

c) Write the linearized state space equations around $\vec{x}_e$ and $\vec{u}_e$. Convert it into the following form and find A and B.

$$\frac{d}{dt}\vec{x}(t) = A(\vec{x} - \vec{x}_e) + B(\vec{u} - \vec{u}_e)$$

d) Prove that the linearized model is controllable for every $\alpha, \beta > 0$.

HINT: A system is controllable iff matrix $C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is full rank and n is the number of states. In our system since $n = 2$ we have $C = \begin{bmatrix} B & AB \end{bmatrix}$ full rank

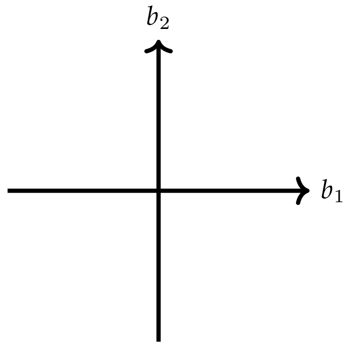
5 Discrete Time Control

Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

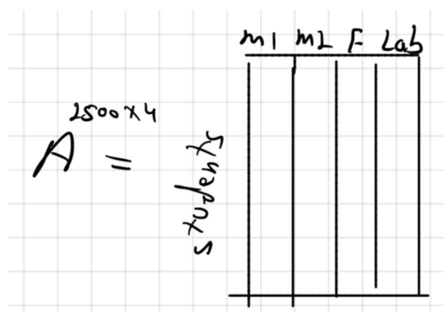
a) Determine if the system is stable.

b) Determine the set of all (b_1, b_2) values for which the system is **not** controllable and sketch this set of points in the b_1-b_2 plane below.



6 PCA Midterm question

Consider a matrix $A \in \mathbb{R}^{2500\times 4}$ which represents the EE16B Sp'2025 midterm 1, midterm 2, final and lab grades for all 2500 students taking the class.

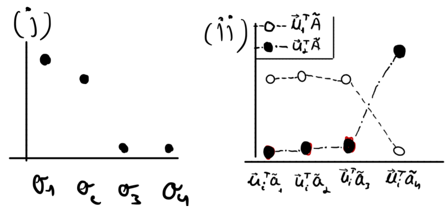


To perform PCA, you subtract the mean of each column and store the results in $\tilde{A}$. Your analysis includes:

a) Computing the SVD: $\tilde{A} = \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T + \sigma_3\vec{u}_3\vec{v}_3^T + \sigma_4\vec{u}_4\vec{v}_4^T$ and plot the singular values.

b) Computing the graph $\vec{u}_1^T\tilde{A}$ and $\vec{u}_2^T\tilde{A}$

The analysis data are plotted below:



Based on the analysis, answer the following true or false questions. Briefly explain your answer.

a) The data can be approximated well by two principle components.

b) The students' exam scores have significant correlation between the exams.

c) The middle plot (ii) shows that students who did well on the exam did not do well in the labs and vice versa.

d) One of the principle components attributes is solely associated with lab scores and not with exam scores.