1 Fundamental Theorem of Solutions to Differential Equations

In this question, you will discover the power of the fundamental theorem of solutions to differential equations. For convenience, we shall restate the theorem here.

Theorem. Consider a differential equation of the form,

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n}(t) + \alpha_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}(t) + \ldots + \alpha_1 \frac{\mathrm{d}y}{\mathrm{d}t}(t) + \alpha_0 y(t) = 0$$

Given n initial conditions of the form,

$$y(t_0) = a_0, \frac{\mathrm{d}y}{\mathrm{d}t}(t_0) = a_1, \ldots, \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}(t_0) = a_{n-1},$$

there exists a unique solution (say, f).

a) Consider the following 2 functions.

$$\phi_1(x) = e^x, \phi_2(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Prove that $\phi_1(x) = \phi_2(x)$ by showing that both functions satisfy the following differential equation:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = f(x) \text{ with } f(0) = 1$$

Side note: Assume $0^0 = 1$.

b) Consider the following 2 functions.

$$\phi_1(x) = \cos(x), \phi_2(x) = \cos(-x)$$

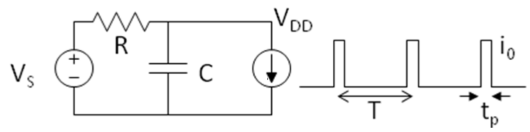
Prove that $\phi_1(x) = \phi_2(x)$ by showing that both functions satisfy the following differential equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) = -f(x) \text{ with } f(0) = 1, \frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0$$

2 IC Power Supply

Digital integrated circuits (ICs) often have very non-uniform current requirements which can cause voltage noise on the supply lines. If one IC is adding a lot of noise to the supply line, it can affect the performance of other ICs that use the same power supply, which can hinder performance of the entire device. For this reason, it is important to take measures to mitigate, or "smooth out", the power supply noise that each IC creates. A common way of doing this is to add a "supply capacitor" between each IC and the power supply. (If you look at a circuit board, and the supply capacitor is the small capacitor next to each IC.)

Here's a simple model for a power supply and digital circuit:



The current source is modeling the "spiky," non-uniform nature of digital circuit current consumption. The resistor represents the sum of the source resistance of the supply and any wiring resistance between the supply and the load.

The capacitor is added to try to minimize the noise on V_{DD}. Assuming that $V_s = 3\text{V}$, $R = 1\Omega$, $i_0 = 1\text{A}$, $T = 10\text{ns}$, and $t_p = 1\text{ns}$,

a) Sketch the voltage V_{DD} vs. time for one or two periods T assuming that $C = 0$.

b) Give expressions for and sketch the voltage V_{DD} vs. time for one or two periods T for each of three different capacitor values for C: 1pF , 1nF, 1μF. (1pF = 10^{-12}F, 1nF = 10^{-9}F, 1μF = 10^{-6}F)

c) Launch the attached Jupyter notebook to interact with a simulated version of this IC power supply. Try to simulate the scenarios outlined in the previous parts. For one of these scenarios, keep the RC time constant fixed, but vary the relative value of R vs. C (e.g. compare $R = 1, C = 2e-9$ to the case where $R = 2, C = 1e-9$). **Is it better to have a lower R or lower C value for a fixed RC time constant when attempting to minimize supply noise? Give an intuitive explanation for why this might be the case.**

3 Simple scalar differential equations driven by an input

In class, you learned that the solution for $t \geq 0$ to the simple scalar first-order differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{1}$$

with initial condition

$$x(t = 0) = x_0 \tag{2}$$

is given for $t \geq 0$ by

$$x(t) = x_0 e^{\lambda t}. \tag{3}$$

In an earlier homework, you proved that these solutions are *unique*– that is, that $x(t)$ of the form in (3) are the only possible solutions to the equation (1) with the specified initial condition (2).

In this question, we will extend our understanding to differential equations with inputs.

In particular, the scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) \tag{4}$$

where $u(t)$ is a known function of time from $t = 0$ onwards.

a) Suppose that you are given an $x_g(t)$ that satisfies both (2) and (4) for $t \geq 0$.

Show that if $y(t)$ also satisfies (2) and (4) for $t \geq 0$, then it must be that $y(t) = x_g(t)$ for all $t \geq 0$.

(HINT: You already used ratios in an earlier HW to prove that two things were necessarily equal. This time, you might want to use differences. Be sure to leverage what you already proved earlier instead of having to redo all that work.)

b) Suppose that the given $u(t)$ starts at $t = 0$ (it is zero before that) and is a nicely integrable function (feel free to assume bounded and continuously differentiably with bounded derivative — whatever conditions you assumed in your calculus course when considering integration and the fundamental theorem of calculus). Let

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \tag{5}$$

for $t \geq 0$.

Show that the $x_c(t)$ defined in (5) indeed satisfies (4) and (2).

Note: the τ here in (5) is just a dummy variable of integration. We could have used any letter for that local variable. We just used τ because it visually reminds us of t while also looking different. If you think they look too similar in your handwriting, feel free to change the dummy variable of integration to another symbol of your choice.

(HINT: Remember the fundamental theorem of calculus that you proved in your calculus class and manipulate the expression in (5) to get it into a form where you can apply it along with other basic calculus rules.)

c) **Use the previous part to get an explicit expression for $x_c(t)$ for $t \geq 0$ when $u(t) = e^{st}$ for some constant s, when $s \neq \lambda$ and $t \geq 0$.**

d) **Similarly, what is $x_c(t)$ for $t \geq 0$ when $u(t) = e^{\lambda t}$ for $t \geq 0$.**

(HINT: Don't worry if this seems too easy.)

4 Op-Amp Stability

In this question we will revisit the basic op-amp model that was introduced in EE16A and we will add a capacitance C_{out} to make the model more realistic (refer to figure 1). Now that we have the tools to do so, we will study the behavior of the op-amp in positive and negative feedback (refer to figure 2). Furthermore, we will begin looking at the integrator circuit (refer to figure 3) to see how a capacitor in the negative feedback can behave. In the next homework, you will see why it ends up being close to an integrator.

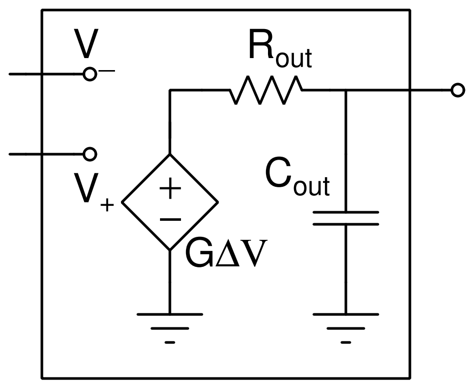
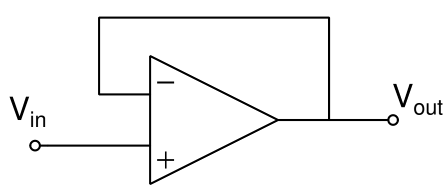
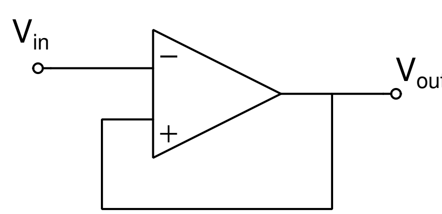


Figure 1: Op-amp model: $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback



(b) "Buffer" in positive feedback that doesn't actually work as a buffer.

Figure 2: Op-amp in buffer configuration

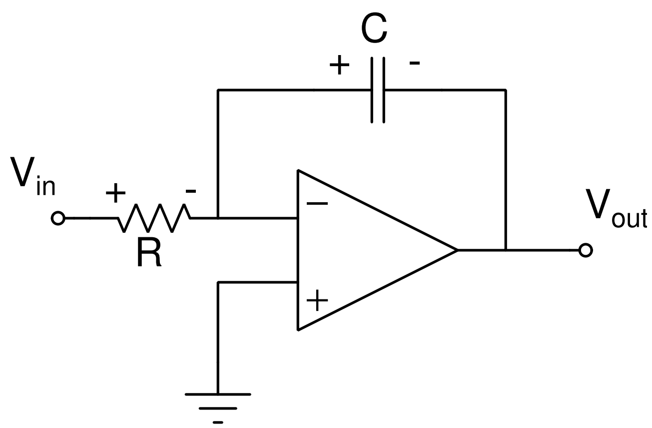


Figure 3: Integrator circuit

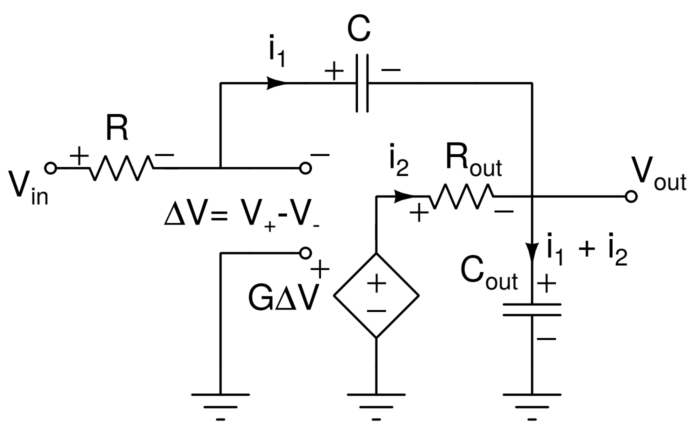


Figure 4: Integrator circuit with Op-amp model

a) Using the op-amp model in figure 1 and the buffer in negative-feedback configuration in figure 2a, **draw a combined circuit**. Remember that $\Delta V = V_+ - V_-$, the voltage difference between the positive and negative labeled input terminals of the op-amp.

   (HINT: Look at figure 4 to see how this was done for the integrator. That might help.)

   Note: here, we have used the Thevenin-equivalent model for the op-amp gain to be compatible with what you have seen in 16A. In more advanced analog circuits courses, it is traditional to use a controlled current source with a resistor in parallel instead.

b) Let's look at the op-amp in negtive feedback. From our discussions in EE16A, we know that the buffer in figure 2a should work with $V_{out} \approx V_{in}$

by the golden rules. **Write a differential-equation for V_{out} by replacing the op-amp with the given model and show what the solution will be as a function of time for a static V_{in}. What does it converge to as $t \to \infty$?** Note: We assume the gain $G > 1$ for all parts of the question.

c) Next, let's look at the op-amp in positive feedback. We know that the configuration given in figure 2b is unstable and V_{out} will just rail. **Again, using the op-amp model in figure 1, show that V_{out} does not converge and hence the output will rail. For positive DC input $V_{in} > 0$, will V_{out} rail to the positive or negative side? Explain.**

d) For an ideal op-amp, we can assume that it has an infinite gain, i.e., $G \to \infty$. **Under these assumptions, show that the op-amp in negative feedback behaves as an ideal buffer, i.e., $V_{out} = V_{in}$.**

5 (OPTIONAL) Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very effective way to really learn material. Having some practice at trying to create problems helps you study for exams much better than simply solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really consolidate your understanding of the course material.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Roughly how many total hours did you work on this homework?**