This homework is designed as a practice for midterm 1.Solutions are published with the homework. Problems 1-3 are designed for midterm practice and problem 4-5 are related to filter and transfer function, materials covered in the class most recently.

There is no self-grade and grading deadline.

1 Matrix Differential Equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \tag{1}$$

where x, y are variables depending on t, $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, and A is a 2×2 matrix with constant coefficients. We call (1) a matrix differential equation.

a) Suppose we have a system of ordinary differential equations

$$x' = 8x + 7y \tag{2}$$

$$y' = -4x - 3y \tag{3}$$

Write this in the form of (1).

b) Compute the eigenvalues of the matrix A from the previous part.

c) We claim that the solution for $x(t), y(t)$ is of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_0 t} + c_3 e^{\lambda_1 t} \end{bmatrix},$$

where c_0, c_1, c_2, c_3 are constants, and λ_0, λ_1 are the eigenvalues of A. Suppose that the initial conditions are $x(0) = 1, y(0) = 1$. Solve for the constants c_0, c_1, c_2, c_3.

d) Verify that the solution for $x(t), y(t)$ found in the previous part satisfies the original system of differential equations (2), (3).

e) We now apply the method above to solve another second-order ordinary differential equation. Suppose we have the system

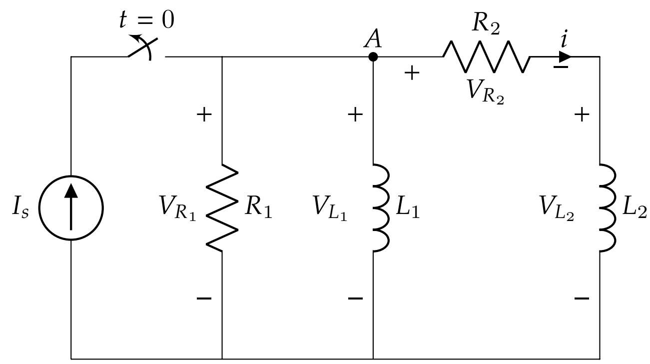
$$z''(t) - 5z'(t) + 6z(t) = 0, \tag{4}$$

where $z' = \frac{dz}{dt}$ and $z'' = \frac{d^2z}{dt^2}$. Write this in the form of (1), by choosing your state variables to be $x(t) = z(t)$, $y(t) = z'(t)$.

f) Solve the system in (4) with the initial conditions $z(0) = 1, z'(0) = 1$, using the method developed in parts (b) and (c).

2 RL Circuit

Consider the circuit below



a) What is $i(0)$.

   Hint: What is the current flowing through L_1 before the switch opens? Consequently what is the current flowing through L_2?

b) What is $\frac{\mathrm{d}i}{\mathrm{d}t}(0)$?

c) What is the relationship between the voltages across L_1 and R_1?

d) Use KCL on node A and the relationship derived above to arrive at a differential equation of the form

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2}(t) + a_1 \frac{\mathrm{d}i}{\mathrm{d}t}(t) + a_0 i(t) = 0$$

   where $i(t)$ is the current going through L_2.

e) Let $R_1 = R_2 = R$ and $L_1 = L_2 = L$. Recall that the above differential equation can be reshaped into the following linear algebra problem:

$$\begin{bmatrix} \frac{\mathrm{d}i}{\mathrm{d}t} \\ \frac{\mathrm{d}^2 i}{\mathrm{d}t^2} \end{bmatrix} = A \begin{bmatrix} i \\ \frac{\mathrm{d}i}{\mathrm{d}t} \end{bmatrix}$$

   What is the A matrix and what are its eigenvalues?

f) Will this circuit exhibit any oscillations?

g) Now, consider the case when the switch is open for time $t < 0$, and the switch closes at $t = 0$. What is V_{R_1}?

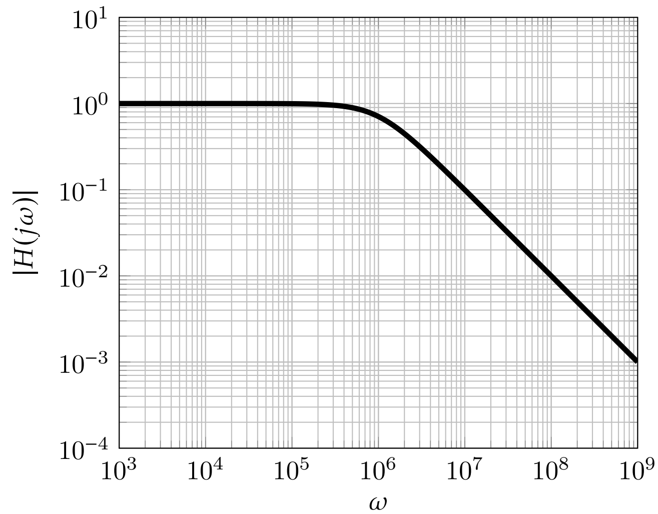
3 Transfer Functions and Filters

a) **Identify each of the Bode Plots, circuits, and transfer functions as either a lowpass or highpass filter.** Indicate your answer by filling in the appropriate bubble.

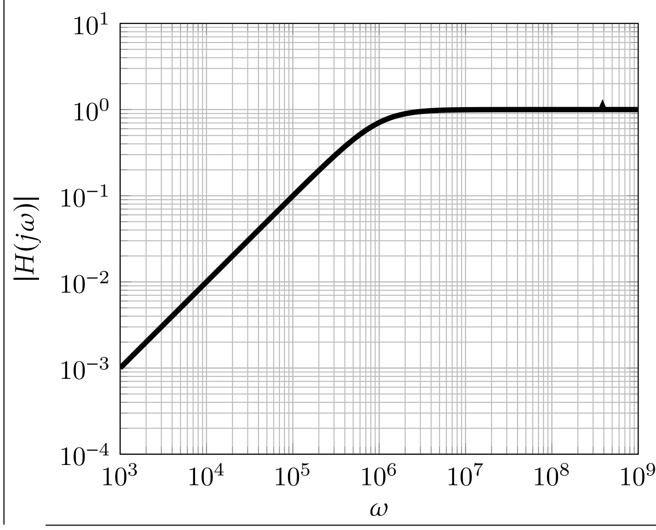
Table 1: Table to be filled in for your answers. Fill in bubbles.

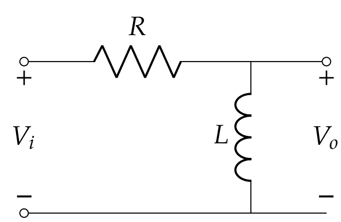
	Lowpass	Highpass
Bode Plot A	○	○
Bode Plot B	○	○
Circuit C	○	○
Circuit D	○	○
Transfer Fn E	○	○
Transfer Fn F	○	○

Bode Plot A ($\omega_c = 10^6$)

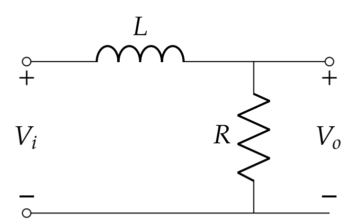


Bode Plot B ($\omega_c = 10^6$)





Circuit C: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$



Circuit D: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$

Transfer function E: $H_E(j\omega) = \frac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}}$ | Transfer function F: $H_F(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_c}}$

b) Consider the three filters in cascade below, with unity-gain op-amp buffers in between them:

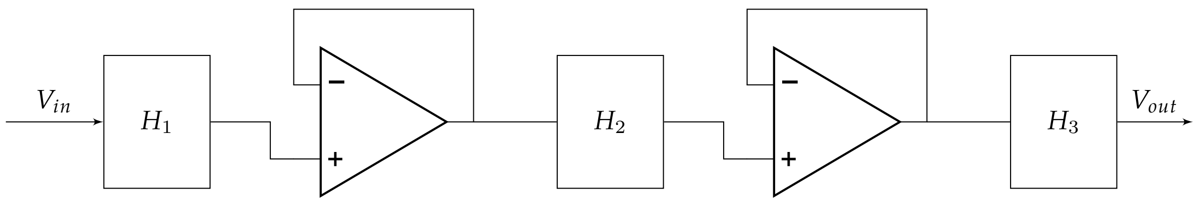


Figure 2: Three filters cascaded via unity-gain op-amp buffers

Suppose that at some frequency ω_0 radians/sec we know that:

$$H_1(j\omega_0) = 3e^{j\frac{\pi}{4}} \quad H_2(j\omega_0) = \frac{1}{2}e^{-j\frac{\pi}{3}} \quad H_3(j\omega_0) = 4e^{j\frac{5\pi}{6}}$$

If $V_{in}(t) = 2\sin\left(\omega_0 t + \frac{\pi}{2}\right)$:

What is the phasor for the input voltage: $\widetilde{V_{in}}$?

What is the phasor for the output voltage: $\widetilde{V_{out}}$?

What is $V_{out}(t)$?

4 Bandpass Filter: Lowpass and Highpass Cascade

In lecture, you heard about how you can go through the design of a bandpass filter by cascading lowpass and highpass filters via buffers (Op-Amps in unity-gain negative feedback to prevent loading effects). In this problem, you will do this for yourself.

Consider an input signal that is composed of the superposition of:

- 20mV level pure tone at 60Hz corresponding to power line noise.
- 1mV level pure tone at 600Hz corresponding to a voice signal.
- 10mV level pure tone at 60kHz corresponding to fluorescent light noise.

This is the signal that you want to filter.

a) We would like to keep the 600Hz tone, which could correspond to a voice signal, for example.

Ignoring any phase offset for each signal (i.e. set the phases to zero), **write the $V_{in}(t)$ that describes the above input in time domain**.

b) **What are the radian/sec frequencies ωs involved and the phasors associated with each tone?**

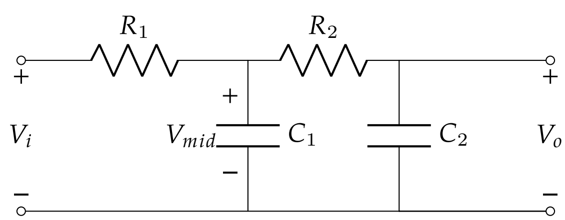
c) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the knee or cutoff-frequency for the lowpass filters**?

d) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the knee or cutoff-frequency for the highpass filters**?

e) Suppose that you only had $1\mu F$ capacitors to use. **What resistance values would you choose for your highpass and lowpass filters so that they have the desired cutoff frequencies?**

5 Transfer functions and why loading is annoying

Consider the circuit below.



The circuit has an input phasor voltage $\widetilde{V}_i$ at frequency ω rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage $\widetilde{V}_o$ at output terminals.

a) We are going to construct the transfer function $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$ in two steps.

We will compute two intermediate transfer functions, $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$ and $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$. Then, we will find the overall transfer function as the product of these two intermediate transfer functions, i.e. $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$.

This approach is valid since the $\widetilde{V}_{mid}$ cancel.

For the first step, **find the intermediate transfer function $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$.** Have your expression be in terms of Z_{R2} and Z_{C2}, that is the impedances of R_2 and C_2.

b) Now, **compute the other intermediate transfer function** $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$. Have your expression be in terms of Z_{R1}, Z_{R2}, Z_{C1}, and Z_{C2}. (i.e. Don't forget to consider the impact of loading by R_2 and C_2 in this transfer function.) *hint: Applying KCL at the V_{mid} node would be a good place to start. You should try to find an expression for H_1 that has factors that H_2 can cancel out.*

c) Then, **use these two intermediate transfer functions to calculate the overall transfer function** as $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$.

d) Sometimes it is useful to collect all the frequency dependence into one place and to figure out how to think about what scale might be somewhat natural for the frequency. **Obtain an expression for** $H(\omega) = \widetilde{V}_o/\widetilde{V}_i$ **in the form**

$$H(\omega) = \frac{V_o}{V_i} = \frac{1}{1 + 2\xi\frac{j\omega}{\omega_c} + \frac{(j\omega)^2}{\omega_c^2}},$$

given that $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, $C_1 = \frac{9}{2}\,\text{F}$, and $C_2 = 1\,\text{F}$. What are the values of ξ and ω_c?

e) **For the previous case, what is the magnitude of the transfer function at the $\omega = \omega_c$ you calculated?**

This is here so that you can see that just because we called it ω_c doesn't mean that the amplitude here is $\frac{1}{\sqrt{2}}$.

f) We can express the transfer function $H(\omega)$ in the polar form. That is,

$$H(\omega) = M(\omega)e^{j\phi(\omega)}$$

The functions $M(\omega)$ and $\phi(\omega)$ are the magnitude and the phase angle of $H(\omega)$, respectively. **Write down $M(\omega)$ and $\phi(\omega)$ using the transfer function you derived in part (b).**

g) Use a computer and then **draw Bode Plots of $|H(\omega)|$ and $\angle H(\omega)$.**