1 Fundamental Theorem of Solutions to Differential Equations

In this question, you will discover the power of the fundamental theorem of solutions to differential equations. For convenience, we shall restate the theorem here.

Theorem. Consider a differential equation of the form,

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n}(t) + \alpha_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}(t) + \ldots + \alpha_1 \frac{\mathrm{d}y}{\mathrm{d}t}(t) + \alpha_0 y(t) = 0$$

Given n initial conditions of the form,

$$y(t_0) = a_0, \frac{\mathrm{d}y}{\mathrm{d}t}(t_0) = a_1, \ldots, \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}(t_0) = a_{n-1},$$

there exists a unique solution (say, f).

a) Consider the following 2 functions.

$$\phi_1(x) = e^x, \phi_2(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Prove that $\phi_1(x) = \phi_2(x)$ by showing that both functions satisfy the following differential equation:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = f(x) \text{ with } f(0) = 1$$

Side note: Assume $0^0 = 1$.

Solution

Let's first look at $\phi_1(x)$.

$$\phi_1(x = 0) = e^0 = 1$$
$$\frac{d\phi_1}{dx}(x) = e^x$$

So ϕ_1 satisfies the conditions. Now let's look at $\phi_2(x)$

$$\phi_2(x = 0) = \sum_{n=0}^{\infty} \frac{0^n}{n!}$$

Removing the first term from the series, we get

$$\phi_2(x = 0) = \frac{0^0}{1} + \sum_{n=1}^{\infty} \frac{0^n}{n!}$$

Zero to any positive finite power is zero, so we can say

$$\sum_{n=1}^{\infty} \frac{0^n}{n!} = 0$$
$$\phi_2(x = 0) = 0^0 = 1$$

To find the derivative of $\phi_2(x)$, we can say

$$\frac{d}{dx} \sum_{n=0}^{\infty} f(x) = \sum_{n=0}^{\infty} \frac{df}{dx}(x)$$
$$\frac{df}{dx}(x) = \frac{d}{dx}(\frac{x^n}{n!}) = \frac{nx^{n-1}}{n!}$$
$$\frac{d\phi_2}{dx}(x) = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!}$$

If we remove the first term of the series, we get

$$\frac{d\phi_2}{dx}(x) = \frac{0x^{-1}}{1} + \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!}$$
$$\frac{d\phi_2}{dx}(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!}$$

Using the fact that

$$\frac{n}{n!} = \frac{1}{(n-1)!}$$

We can say

$$\frac{d\phi_2}{dx}(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

If we use substitution and let $k = (n-1)$, we get

$$\frac{d\phi_2}{dx}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \phi_2(x)$$

Both ϕ_1 and ϕ_2 meet the conditions, therefore they must be the same function.

b) Consider the following 2 functions.

$$\phi_1(x) = \cos(x), \phi_2(x) = \cos(-x)$$

Prove that $\phi_1(x) = \phi_2(x)$ by showing that both functions satisfy the following differential equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) = -f(x) \text{ with } f(0) = 1, \frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0$$

Solution

Initial conditions:

$$\cos(0) = \cos(-0) = 1$$

So both ϕ_1 and ϕ_2 satisfy $f(0) = 1$. Taking the first derivative of each:

$$\frac{d\phi_1}{dx} = -\sin(x)$$
$$\frac{d\phi_2}{dx} = \sin(-x)$$

Plugging in 0 for x:

$$-\sin(0) = 0$$
$$\sin(-0) = 0$$

Both functions satisfy $\frac{df}{dx}(0) = 0$. Taking the second derivative of each:

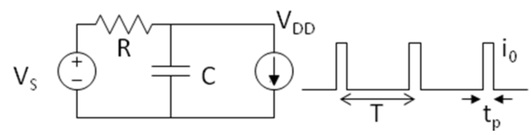
$$\frac{d^2\phi_1}{dx^2}(x) = -\cos(x) = -\phi_1(x)$$
$$\frac{d^2\phi_2}{dx^2}(x) = -\cos(-x) = -\phi_2(x)$$

Both functions satisfy all three conditions, so $\phi_1(x) = \phi_2(x)$.

2 IC Power Supply

Digital integrated circuits (ICs) often have very non-uniform current requirements which can cause voltage noise on the supply lines. If one IC is adding a lot of noise to the supply line, it can affect the performance of other ICs that use the same power supply, which can hinder performance of the entire device. For this reason, it is important to take measures to mitigate, or "smooth out", the power supply noise that each IC creates. A common way of doing this is to add a "supply capacitor" between each IC and the power supply. (If you look at a circuit board, and the supply capacitor is the small capacitor next to each IC.)

Here's a simple model for a power supply and digital circuit:



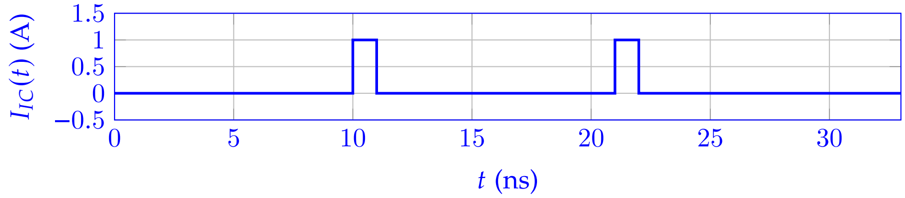
The current source is modeling the "spiky," non-uniform nature of digital circuit current consumption. The resistor represents the sum of the source resistance of the supply and any wiring resistance between the supply and the load.

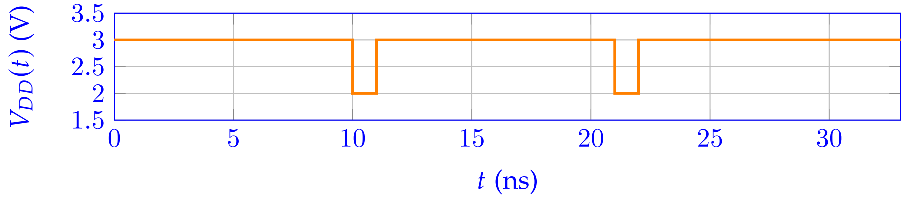
The capacitor is added to try to minimize the noise on V_{DD}. Assuming that $V_s = 3\text{V}$, $R = 1\Omega$, $i_0 = 1\text{A}$, $T = 10\text{ns}$, and $t_p = 1\text{ns}$,

a) Sketch the voltage V_{DD} vs. time for one or two periods T assuming that $C = 0$.

Solution

If $C = 0$, then this circuit will respond instantaneously to changes in the current; thus we may break this down into two segments, wherein the current source $I_{IC}(t)$ equals 0 (and thus $V_{dd} = V_s$), and where the current source $I_{IC}(t)$ equals i_0 (and thus $V_{dd} = V_s - i_0 R = 2\text{V}$). These will follow the current source's flips precisely. With that in mind, your sketch should look something like this:





In the above sketch, we have the first current spike at $t = 10$ ns. Yours doesn't have to align with that: for example, if you had the first current spike at $t = 0$, that's okay. However, what *does* matter is that you have the timing between the current spikes drawn correctly.

b) Give expressions for and sketch the voltage V_{DD} vs. time for one or two periods T for each of three different capacitor values for C: 1pF , 1nF, 1μF. (1pF = 10^{-12}F, 1nF = 10^{-9}F, 1μF = 10^{-6}F)

Solution

Since the current through the source is a series of pulses, it will be easiest if we solve for $V_{dd}(t)$ assuming a piecewise constant I_C. Starting with KVL:

$$V_S = V_R + V_{dd} \tag{1}$$

$$V_S = (I_C + C\frac{d}{dt}V_{dd})R + V_{dd} \tag{2}$$

$$\frac{d}{dt}V_{dd} = \frac{1}{RC}(V_S - RI_C - V_{dd}) \tag{3}$$

Where I_C is the piecewise constant value of the current flowing through the digital circuit. At this point we can use substitution with $\widetilde{V} = (V_S - RI_C - V_{dd})$.

$$\frac{d}{dt}\widetilde{V} = -\frac{d}{dt}V_{dd} \tag{4}$$

$$\frac{d}{dt}V_{dd} = \frac{\widetilde{V}}{RC} \tag{5}$$

$$\frac{d}{dt}\widetilde{V} = -\frac{\widetilde{V}}{RC} \tag{6}$$

$$\widetilde{V}(t) = Ae^{-\frac{t}{RC}} \tag{7}$$

Substituting back to solve for $V_{dd}(t)$ we get the following general expression for the voltage during any one piecewise constant time slice:

$$V_{dd}(t) = V_S - RI_C - Ae^{-\frac{t}{RC}} \tag{8}$$

From here, based on the value of the piecewise current I_C and the initial conditions imposed by previous time segments we can simplify the V_{dd} expression and solve for A. At the start, we will assume a convenient initial condition which corresponds with the behavior of V_{dd} if $I_C = 0$ for a long time before $t = 0$. In this case, $V_{dd} = V_S$ and there is zero current flowing through the circuit. Things get exciting once the first current pulse starts such that $I_C = i_0$. The initial condition for this piecewise section is the final voltage V_{dd} from the previous section, V_S.

$$V_S = V_S - Ri_0 - Ae^0 \tag{9}$$

$$A = -Ri_0 \tag{10}$$

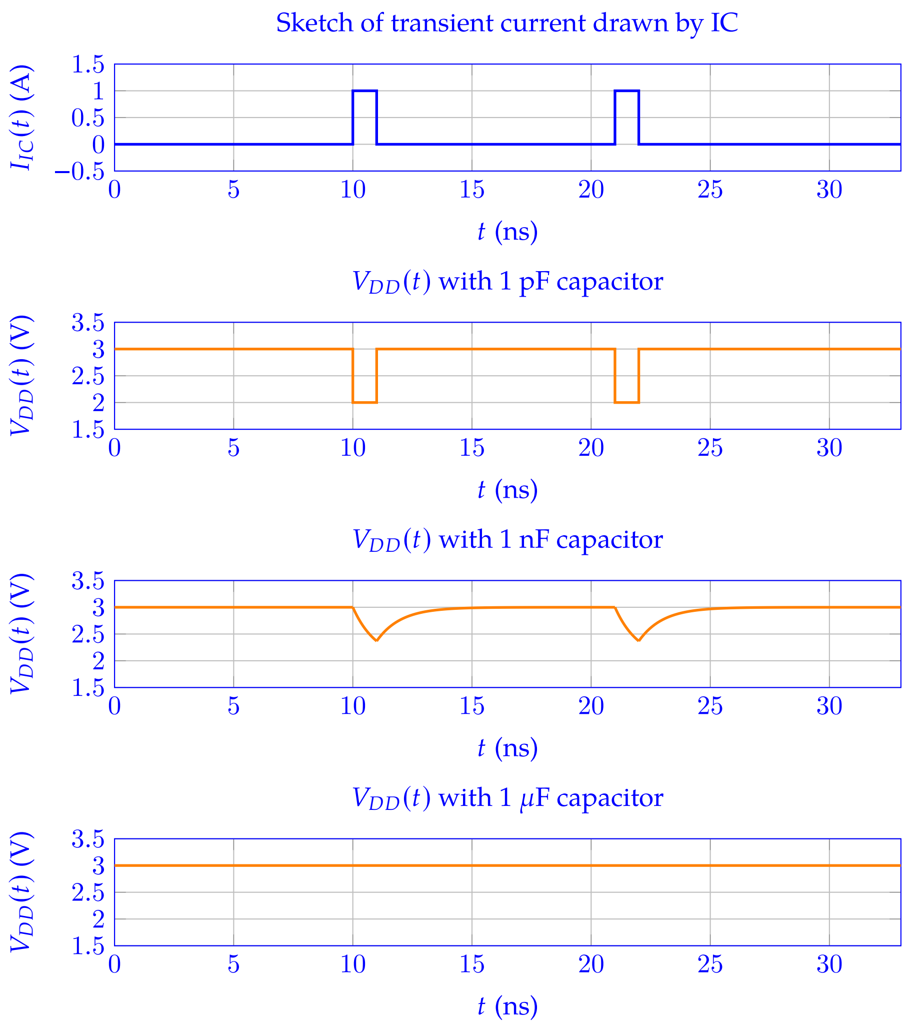
$$V_{dd}(t) = V_S - Ri_0(1 - e^{-\frac{t}{RC}}) \tag{11}$$

You can compute the voltage at the end of the pulse by plugging in $t = t_p$ and the R and C values for your scenario. This voltage will serve as the initial condition for the next piecewise constant section. The process of simplifying the general piecewise differential equation and solving for A can be performed repeatedly to determine the shape of the plot for further pulses.

In general: each of the three curves will tend towards a final value $V_{dd} = V_s$, growing exponentially slower towards this goal as time progresses. However, on each time interval t_p, the current source will start drawing charge from both V_s–whose current decreases as time proceeds–and C–whose charge, and therefore whose potential to contribute voltage, tends to increase with time. With each t_p, V_{dd} decreases nonlinearly, as there are both exponential and linear factors contributing to the rise and fall of the voltage.

With a lower capacitance, we will see the capacitor charge and discharge faster with time–this means that V_{dd} will fluctuate more/change more drastically on t_p; as you increase capacitance, this fluctuation is less evident, as C has more charge to pull from, and will thus be less affected by the change of charge incurred by the current source.

The final sketches should look something like this:



Note that if your solutions contain just the correct plots without much verbose, textual explanation of the plots, then you still deserve full credit.

The idea here is to see the effect that the capacitors have on $V_{DD}(t)$ when viewed at the time scale of the current spikes.

- the 1 pF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1\text{ps}$, that is $1\text{picosecond} = 10^{-12}$ seconds, and the effect that this has on $V_{DD}(t)$ is invisible at the nanosecond time scale. For this reason, we can conclude that the 1 pF capacitor would not be adequate to mitigate the noise that the IC will put on the power supply.

- the 1 nF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1\text{ns}$. This is a long enough time scale that the effect on $V_{DD}(t)$ will be visible. At the end of the 1 ns current spike, $V_{DD}(t)$ will have dropped from 3 V to $2 + \exp(-1) \approx 2.37$ V. This means that the 1 nF capacitor is actually reducing the power supply noise a little bit, but not much.

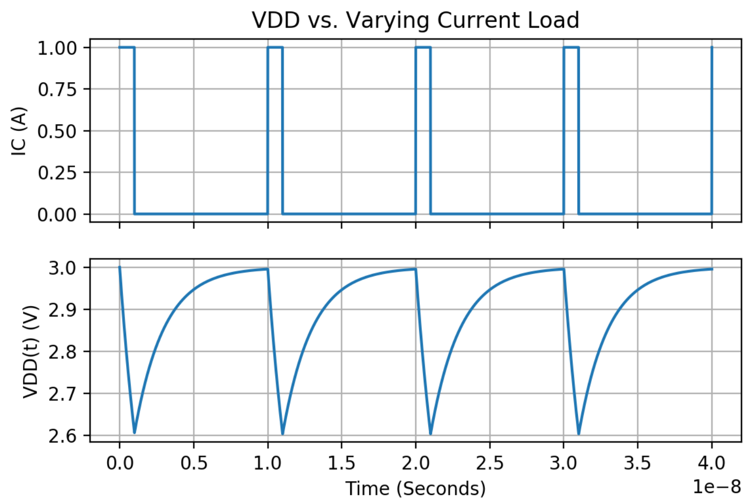
- the 1 μF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1\mu\text{s}$. This time constant is 1000 times longer than the duration of the current spike. At the end of the current spike, $V_{DD}(t)$ will have dropped by only one millivolt, so at the scale at which these sketches are drawn, there is no visible change. The 1 μF capacitor has almost totally removed the power supply noise.

c) Launch the attached Jupyter notebook to interact with a simulated version of this IC power supply. Try to simulate the scenarios outlined in the previous parts. For one of these scenarios, keep the RC time constant fixed, but vary the relative value of R vs. C (e.g. compare $R = 1, C = 2e - 9$ to the case where $R = 2, C = 1e - 9$). **Is it better to have a lower R or lower C value for a fixed RC time constant when attempting to minimize supply noise? Give an intuitive explanation for why this might be the case.**

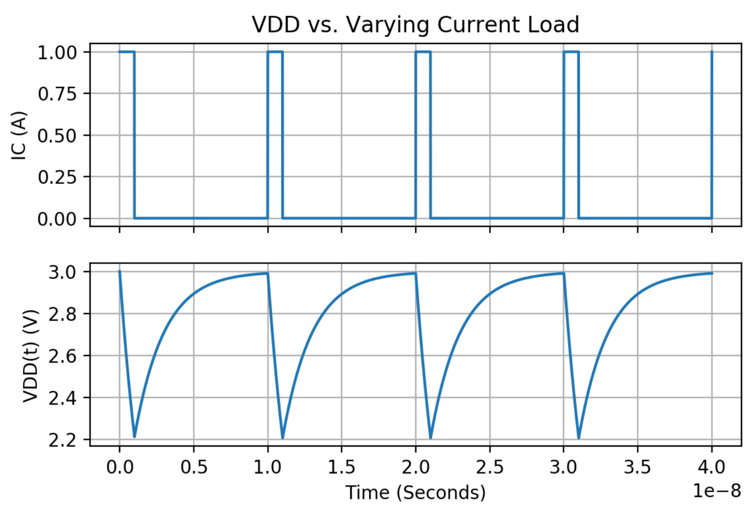
Solution

A lower resistance and higher capacitance leads to smaller variation in the supply voltage with each current spike. One intuitive way to see this is to think about where the charge comes from whenenver the current source turns on. The charge comes from the capacitor and from the voltage source through the resistor. By $Q = CV$ for a constant amount of charge drawn, a larger capacitor results in lower voltage change. By $V = IR$ for a constant amount of current drawn through the resistor, a larger resistor leads to a larger voltage drop.

In the case where $R = 1$ and $C = 2e - 9$, we get the following plot:



In the case where $R = 2$ and $C = 1e - 9$, we get the following plot:



Notice that the shape of the V_{dd} curves is the same becuase the RC constant is the same. However they drop to different voltages by the end of each pulse.

3 Simple scalar differential equations driven by an input

In class, you learned that the solution for $t \geq 0$ to the simple scalar first-order differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{12}$$

with initial condition

$$x(t = 0) = x_0 \tag{13}$$

is given for $t \geq 0$ by

$$x(t) = x_0 e^{\lambda t}. \tag{14}$$

In an earlier homework, you proved that these solutions are *unique*– that is, that $x(t)$ of the form in (14) are the only possible solutions to the equation (12) with the specified initial condition (13).

In this question, we will extend our understanding to differential equations with inputs.

In particular, the scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) \tag{15}$$

where $u(t)$ is a known function of time from $t = 0$ onwards.

a) Suppose that you are given an $x_g(t)$ that satisfies both (13) and (15) for $t \geq 0$.

   Show that if $y(t)$ also satisfies (13) and (15) for $t \geq 0$, then it must be that $y(t) = x_g(t)$ for all $t \geq 0$.

   (HINT: You already used ratios in an earlier HW to prove that two things were necessarily equal. This time, you might want to use differences. Be sure to leverage what you already proved earlier instead of having to redo all that work.)

   ### Solution

   We had already used ratios in the earlier homework, and so the natural thing to look at is the difference. Consider $z(t) = y(t) - x_g(t)$. We know that

$$\begin{aligned} z(0) &= y(0) - x_g(0) && (16)\\ &= x_0 - x_0 && (17)\\ &= 0. && (18) \end{aligned}$$

Furthermore,

$$\frac{d}{dt}z(t) = \frac{d}{dt}y(t) - \frac{d}{dt}x_g(t) \tag{19}$$
$$= \lambda y(t) + u(t) - (\lambda x_g(t) + u(t)) \tag{20}$$
$$= \lambda(y(t) - x_g(t)) + (u(t) - u(t)) \tag{21}$$
$$= \lambda z(t) \tag{22}$$

By (22) and (18), we know that $z(t)$ satisfies the exact conditions for which we proved uniqueness in an earlier homework. (This is why we cared so much about getting the zero initial condition case correct without handwaving.) So we know that $z(t) = 0e^{\lambda t} = 0$ for all $t \geq 0$.

This means that $y(t) - x_g(t) = 0$ and hence $y(t) = x_g(t)$. This successfully proves the uniqueness of solutions to simple scalar differential equations with inputs.

b) Suppose that the given $u(t)$ starts at $t = 0$ (it is zero before that) and is a nicely integrable function (feel free to assume bounded and continuously differentiably with bounded derivative — whatever conditions you assumed in your calculus course when considering integration and the fundamental theorem of calculus). Let

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \tag{23}$$

for $t \geq 0$.

Show that the $x_c(t)$ defined in (23) indeed satisfies (15) and (13).

Note: the τ here in (23) is just a dummy variable of integration. We could have used any letter for that local variable. We just used τ because it visually reminds us of t while also looking different. If you think they look too similar in your handwriting, feel free to change the dummy variable of integration to another symbol of your choice.

(HINT: Remember the fundamental theorem of calculus that you proved in your calculus class and manipulate the expression in (23) to get it into a form where you can apply it along with other basic calculus rules.)

Solution

Checking (13) just involves plugging in $t = 0$ into (23).

$$x_c(0) = x_0 e^0 + \int_0^0 u(\tau) e^{\lambda(0-\tau)} d\tau = x_0 + 0 = x_0. \tag{24}$$

The real action is in checking that this satisfies the differential equation with the given input. To do this, there are many approaches. One would be to use the big hammer of the full Fundamental Theorem of Calculus in Leibniz form. However, in this case, we can do this in a simpler way by first simplifying the integral by pulling out factors that do not vary with the variable of integration. Notice that:

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \tag{25}$$
$$= x_0 e^{\lambda t} + e^{\lambda t} \int_0^t u(\tau) e^{-\lambda \tau} d\tau. \tag{26}$$

Taking a derivative we have:

$$\frac{d}{dt} x_c(t) = \lambda x_0 e^{\lambda t} + \lambda e^{\lambda t} \int_0^t u(\tau) e^{-\lambda\tau} d\tau + e^{\lambda t} \frac{d}{dt} \int_0^t u(\tau) e^{-\lambda\tau} d\tau \tag{27}$$
$$= \lambda x_c(t) + e^{\lambda t} \frac{d}{dt} \int_0^t u(\tau) e^{-\lambda\tau} d\tau \tag{28}$$
$$= \lambda x_c(t) + e^{\lambda t} u(t) e^{-\lambda t} \tag{29}$$
$$= \lambda x_c(t) + u(t) \tag{30}$$

where we used the product rule in (27) and the basic fundamental theorem of calculus in (29).

Because we have checked that the solution satisfies the conditions of the differential equation, and we have shown uniqueness in the previous part, we know that this must be the only solution.

The discussion and the supported course notes show how an argument building from approximating by piecewise constant functions and then Reimann integration lets you naturally guess this form of the solution.

c) **Use the previous part to get an explicit expression for $x_c(t)$ for $t \geq 0$ when $u(t) = e^{st}$ for some constant s, when $s \neq \lambda$ and $t \geq 0$.**

Solution

In this part we are given input $u(t) = e^{st}$. We have to solve a differential equation of the form:

$$\frac{d}{dt} x(t) = \lambda x(t) + e^{st} \tag{31}$$

In the previous part we were given equation (23) to solve for $x_c(t)$ for $t \geq 0$, for a nonhomogeneous ode of the form (15) with any input $u(t)$.

Plugging into equation (23) we get:

$$
\begin{aligned}
x_c(t) &= x_0 e^{\lambda t} + \int_0^t e^{s\tau} e^{\lambda(t-\tau)} d\tau && (32)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \int_0^t e^{\tau(s-\lambda)} d\tau && (33)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \Big[\frac{1}{s-\lambda} e^{(s-\lambda)\tau}\Big]_{\tau=0}^{\tau=t} && (34)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \Big[\frac{1}{s-\lambda} e^{(s-\lambda)t} - \frac{1}{s-\lambda} e^{(s-\lambda)0}\Big] && (35)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \Big[\frac{1}{s-\lambda} e^{(s-\lambda)t} - \frac{1}{s-\lambda}\Big] && (36)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \frac{1}{s-\lambda} e^{(s-\lambda)t} - e^{\lambda t} \frac{1}{s-\lambda} && (37)\\
&= x_0 e^{\lambda t} + \frac{1}{s-\lambda} e^{st} - \frac{1}{s-\lambda} e^{\lambda t} && (38)\\
&= (x_0 - \frac{1}{s-\lambda}) e^{\lambda t} + \frac{e^{st}}{s-\lambda} && (39)
\end{aligned}
$$

Thus we get $x_c(t) = (x_0 - \frac{1}{s-\lambda})e^{\lambda t} + \frac{e^{st}}{s-\lambda}$.

d) **Similarly, what is $x_c(t)$ for $t \geq 0$ when $u(t) = e^{\lambda t}$ for $t \geq 0$.**

(HINT: Don't worry if this seems too easy.)

Solution

Similar to the previous part here we are given another exponential e^{st} as an input. However here $s = \lambda$. Just like the previous part we will proceed by plugging our new input into (15):

$$
\begin{aligned}
x_c(t) &= x_0 e^{\lambda t} + \int_0^t e^{\lambda \tau} e^{\lambda(t-\tau)} d\tau && (40)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \int_0^t e^{\tau(\lambda-\lambda)} d\tau && (41)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \int_0^t 1 d\tau && (42)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \big[\tau\big]_{\tau=0}^{\tau=t} && (43)\\
&= x_0 e^{\lambda t} + t e^{\lambda t} && (44)
\end{aligned}
$$

Thus we get $x_c(t) = x_0 e^{\lambda t} + t e^{\lambda t}$.

This basic pattern can be continued. We can plug in $te^{\lambda t}$ as an input and get further polynomials in t multiplying the same exponential.

4 Op-Amp Stability

In this question we will revisit the basic op-amp model that was introduced in EE16A and we will add a capacitance C_{out} to make the model more realistic (refer to figure 1). Now that we have the tools to do so, we will study the behavior of the op-amp in positive and negative feedback (refer to figure 2). Furthermore, we will begin looking at the integrator circuit (refer to figure 3) to see how a capacitor in the negative feedback can behave. In the next homework, you will see why it ends up being close to an integrator.

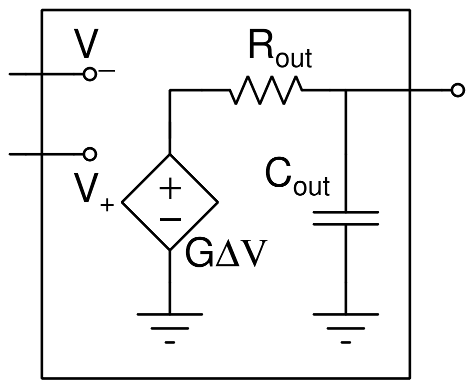
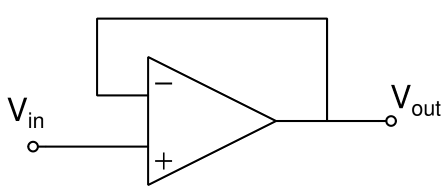
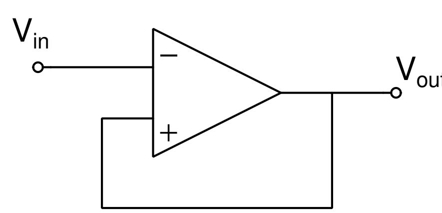


Figure 1: Op-amp model: $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback



(b) "Buffer" in positive feedback that doesn't actually work as a buffer.

Figure 2: Op-amp in buffer configuration

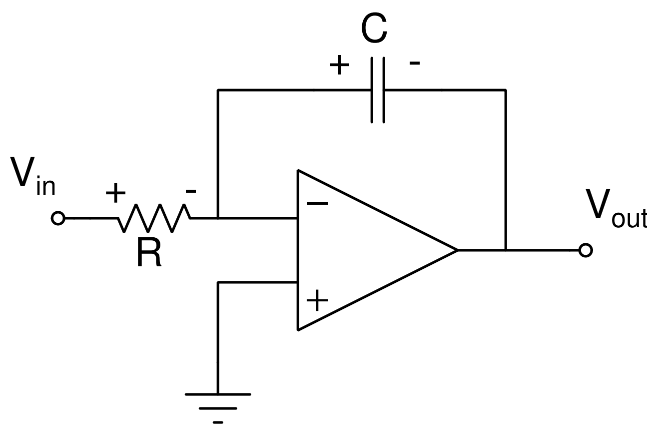


Figure 3: Integrator circuit

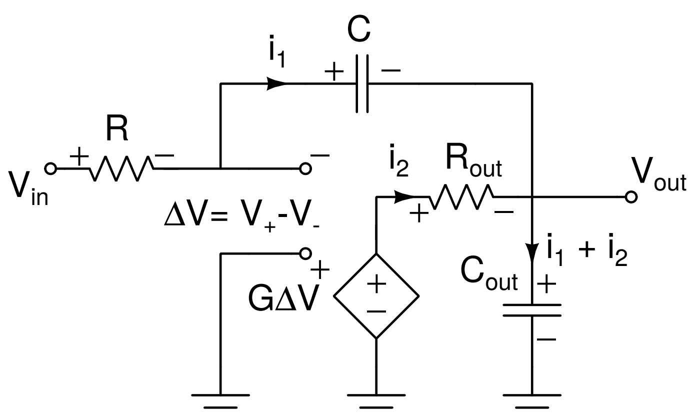


Figure 4: Integrator circuit with Op-amp model

a) Using the op-amp model in figure 1 and the buffer in negative-feedback configuration in figure 2a, **draw a combined circuit**. Remember that $\Delta V = V_+ - V_-$, the voltage difference between the positive and negative labeled input terminals of the op-amp.

(HINT: Look at figure 4 to see how this was done for the integrator. That might help.)

Note: here, we have used the Thevenin-equivalent model for the op-amp gain to be compatible with what you have seen in 16A. In more advanced analog circuits courses, it is traditional to use a controlled current source with a resistor in parallel instead.

Solution

Please refer to Figure 5 for the completed circuit.

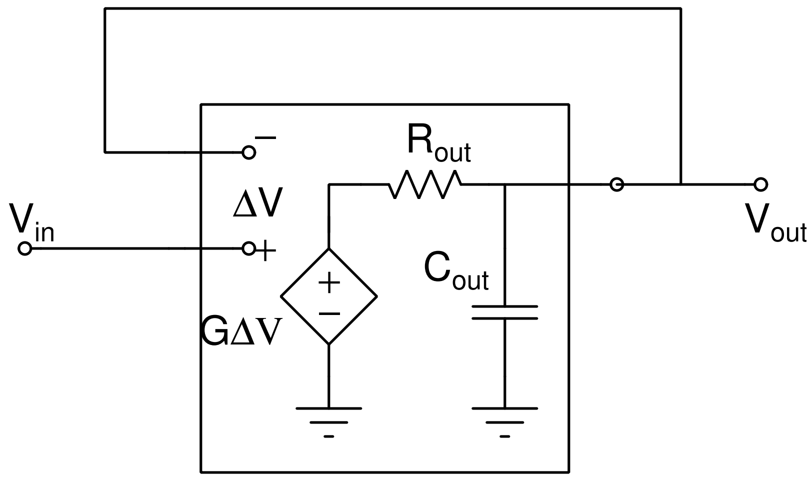


Figure 5: Negative-feedback buffer configuration using given op-amp model

b) Let's look at the op-amp in negtive feedback. From our discussions in EE16A, we know that the buffer in figure 2a should work with $V_{out} \approx V_{in}$ by the golden rules. **Write a differential-equation for V_{out} by replacing the op-amp with the given model and show what the solution will be as a function of time for a static V_{in}. What does it converge to as $t \longrightarrow \infty$?** Note: We assume the gain $G > 1$ for all parts of the question.

Solution

We have $\Delta V = V_{in} - V_{out}$. Next, we can write the following branch equations:

$$i = C_{out} \frac{d}{dt} V_{out}$$
$$G\Delta V = V_R + V_{out}$$
$$G(V_{in} - V_{out}) = RC \frac{d}{dt} V_{out} + V_{out}$$

Simplifying the last line, we get:

$$GV_{in} = RC \frac{d}{dt} V_{out} + (1 + G) V_{out}$$
$$\frac{d}{dt} V_{out} + \frac{1+G}{RC} V_{out} - \frac{G}{RC} V_{in} = 0$$

Solving the above differential equation, with the substitution $\widetilde{V}_{out} = V_{out} - \frac{G}{G+1}V_{in}$, we get

$$\widetilde{V}_{out}(t) = ke^{-\frac{G+1}{RC}t}.$$

Substituting for the initial condition $V_{out}(0) = 0$, we get:

$$V_{out} = \frac{GV_{in}}{G+1}\left(1 - e^{-\frac{G+1}{RC}t}\right).$$

Since $G > 1$, the exponent is negative, hence as $t \to \infty$, the solution will converge to $V_{out} \to \frac{G}{G+1}V_{in}$.

c) Next, let's look at the op-amp in positive feedback. We know that the configuration given in figure 2b is unstable and V_{out} will just rail. **Again, using the op-amp model in figure 1, show that V_{out} does not converge and hence the output will rail. For positive DC input $V_{in} > 0$, will V_{out} rail to the positive or negative side? Explain.**

Solution

This time, we have $\Delta V = V_{out} - V_{in}$. Next, we can write the following branch equations:

$$i = C_{out}\frac{d}{dt}V_{out}$$
$$G\Delta V = V_R + V_{out}$$
$$G(V_{out} - V_{in}) = RC\frac{d}{dt}V_{out} + V_{out}$$

Simplifying the last line, we get:

$$-GV_{in} = RC\frac{d}{dt}V_{out} + (1-G)V_{out}$$
$$\frac{d}{dt}V_{out} + \frac{1-G}{RC}V_{out} + \frac{G}{RC}V_{in} = 0$$

Solving the above differential equation, with the substitution $\widetilde{V}_{out} = V_{out} - \frac{G}{G-1}V_{in}$, we get:

$$\widetilde{V}_{out} = ke^{-\frac{1-G}{RC}t}$$
$$= ke^{\frac{G-1}{RC}t}$$

Substituting for the hypothetical initial condition $V_{out}(0) = 0$, we get:

$$V_{out} = -\frac{GV_{in}}{G-1}\left(e^{\frac{G-1}{RC}t} - 1\right)$$

Since $G > 1$, the exponent is positive, hence as $t \to \infty$, the solution will be unbounded, and $V_{out} \to -\infty$. Of course, it can't grow to negative infinity, and so we can conclude that V_{out} will rail to the negative side if V_{out} had started at zero. The same exact story would hold if V_{out} started anywhere below V_{in}.

The case of V_{out} starting out significantly greater than V_{in} deserves some mention, although not necessary for full credit on this question. In this case, the initial condition for $\widetilde{V}_{out}$ would be positive, and would proceed to unstably attempt to run away to positive infinity. It would be stopped at the positive rail, where it would stay.

Essentially, all that matters is the initial condition of $\widetilde{V}_{out}$ — start out positive, then we rail to a positive rail for V_{out}. Start out negative, then we rail to the negative rail for V_{out}. An op-amp in positive feedback retains its comparator-like character.

d) For an ideal op-amp, we can assume that it has an infinite gain, i.e., $G \to \infty$. **Under these assumptions, show that the op-amp in negative feedback behaves as an ideal buffer, i.e.,** $V_{out} = V_{in}$.

Solution

Taking the limit of our solution in part (a),

$$\begin{aligned} V_{out} &= \lim_{G\to\infty} \frac{GV_{in}}{G+1}\left(1 - e^{-\frac{G+1}{RC}t}\right) \\ &= V_{in} \end{aligned}$$

The coeffecient of the exponent goes to 1 whereas the exponent itself goes to $-\infty$, and hence $1 - e^{-\infty} \to 1$.

5 (OPTIONAL) Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very effective way to really learn material. Having some practice at trying to create problems helps you study for exams much better than simply solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really consolidate your understanding of the course material.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Roughly how many total hours did you work on this homework?**