This homework is due on Wednesday, February 26, 2020, at 11:59PM.

Self-grades are due on Monday, March 2, 2020, at 11:59PM.

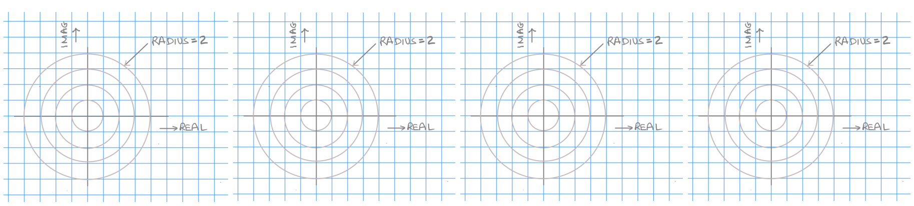
1 Phasors

a) Consider a resistor ($R = 1.5\Omega$), a capacitor ($C = 1$F), and an inductor ($L = 1$H) connected in series. **Give expressions for the impedances of Z_R, Z_C, Z_L for each of these elements as a function of the angular frequency ω.**

Solution

The impedances are as follows: $Z_R = R = 1.5$, $Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega} = -\frac{j}{\omega}$ and $L = j\omega L = j\omega$.

b) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = \frac{1}{2}$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.

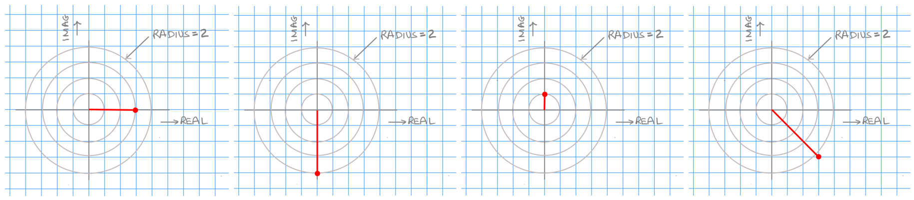


(a) $Z_R(@\omega = 0.5)$ (b) $Z_C(@\omega = 0.5)$ (c) $Z_L(@\omega = 0.5)$ (d) $Z_{total}(@\omega = 0.5)$

Figure 1: Impedances at $\omega = 0.5$.

Solution

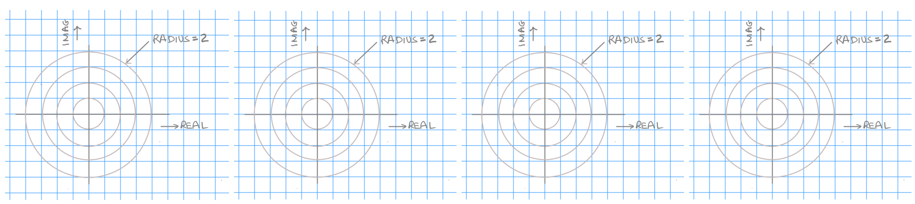
Substituting for $\omega = \frac{1}{2}$ in the above answers, we get, $Z_R = 1.5$, $Z_C = -2j$ and $Z_L = 0.5j$. Since the elements are in series, $Z_{total} = Z_L + Z_C + Z_R = 1.5 - 1.5j$. This has magnitude $1.5\sqrt{2}$ and phase $-\frac{\pi}{4}$. Following are the plots:



(a) $Z_R(@\omega = 0.5)$ (b) $Z_C(@\omega = 0.5)$ (c) $Z_L(@\omega = 0.5)$ (d) $Z_{total}(@\omega = 0.5)$

Figure 2: Impedances at $\omega = 0.5$.

c) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 1$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.

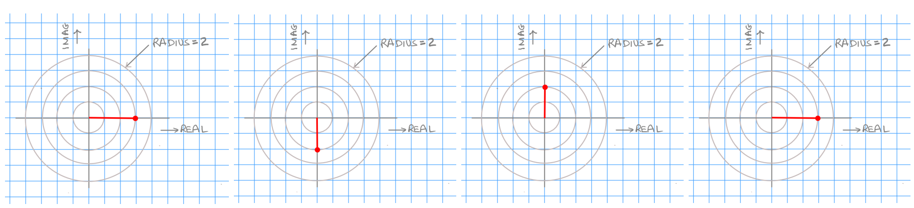


(a) $Z_R(@\omega = 1)$ (b) $Z_C(@\omega = 1)$ (c) $Z_L(@\omega = 1)$ (d) $Z_{total}(@\omega = 1)$

Figure 3: Impedances at $\omega = 1$.

Solution

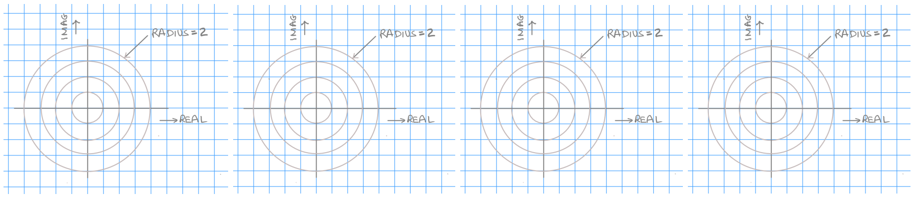
Following the same method as last time, with $\omega = 1$, $Z_R = 1.5$, $Z_C = -j$, $Z_L = j$ and $Z_{total} = 1.5$ Thishas magnitude 1.5 and phase 0.



(a) $Z_R(@\omega = 1)$ (b) $Z_C(@\omega = 1)$ (c) $Z_L(@\omega = 1)$ (d) $Z_{total}(@\omega = 1)$

Figure 4: Impedances at $\omega = 1$.

d) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 2$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.

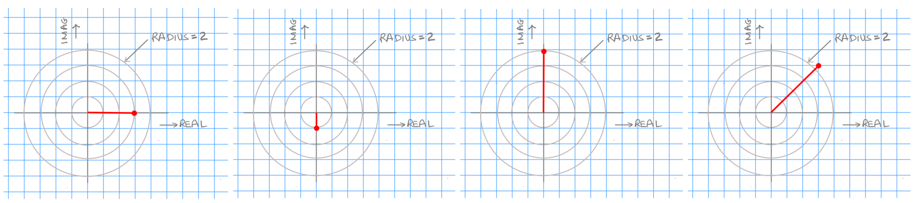


(a) $Z_R(@\omega = 2)$ (b) $Z_C(@\omega = 2)$ (c) $Z_L(@\omega = 2)$ (d) $Z_{total}(@\omega = 2)$

Figure 5: Impedances at $\omega = 2$.

Solution

Again, following the same method as last time, with $\omega = 2$, $Z_R = 1.5$, $Z_C = -0.5j$, $Z_L = 2j$ and $Z_{total} = 1.5 + 1.5j$. This has magnitude $1.5\sqrt{2}$ and phase $+\frac{\pi}{4}$.



(a) $Z_R(@\omega = 2)$ (b) $Z_C(@\omega = 2)$ (c) $Z_L(@\omega = 2)$ (d) $Z_{total}(@\omega = 2)$

Figure 6: Impedances at $\omega = 2$.

e) **For the previous series combination of RLC elements, what is the "natural frequency" ω_n where the series impedance is purely real?**

Solution

From our above answers, clearly the natural frequency, $\omega_n = 1$ rad/s . This is where the imaginary parts of the impedance cancel each other.

2 Low-pass Filter

You have a 1 kΩ resistor and a 1 μF capacitor wired up as a low-pass filter.

a) **Draw the filter circuit, labeling the input node, output node, and ground.**

Solution

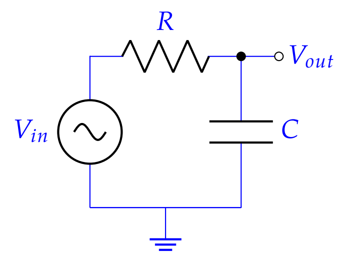


Figure 7: A simple RC circuit

b) **Write down the transfer function of the filter, $H(j\omega)$ that relates the output voltage phasor to the input voltage phasor.** Be sure to use the given values for the components.

Solution

First, we convert everything into the phasor domain. We have,

$$Z_R = R = 1 \times 10^3 \Omega \tag{1}$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega \times 10^{-6}} \mathrm{F} \tag{2}$$

In phasor domain, we can treat these impedances essentially like we treat resistors and recognize the voltage divider. Hence,

$$\widetilde{V}_{out} = \frac{Z_C}{Z_C + Z_R} \widetilde{V}_{in} \tag{3}$$

$$\frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{4}$$

$$= \frac{1}{1 + j\omega RC} \tag{5}$$

$$= \frac{1}{1 + j\omega / \frac{1}{RC}} \tag{6}$$

$$= \frac{1}{1 + j\omega \times 10^{-3}} \tag{7}$$

$$\tag{8}$$

Hence, the corner frequency $\omega_C = \frac{1}{RC} = \frac{1}{10^3 \cdot 10^{-6}} = 10^3$ rad/sec.

c) **Write an exact expression for the *magnitude* of $H(j\omega = j10^6)$, and give an approximate numerical answer.**

Solution

We obtained this expression for the transfer function's magnitude above:

$$|H(\omega)| = \frac{\sqrt{1+\omega^2/\omega_c^2}}{1+\omega^2/\omega_c^2}$$

Plugging in for $\omega = 10^6$:

$$|H(\omega = 10^6)| = \frac{\sqrt{1+(10^6)^2/(10^3)^2}}{1+(10^6)^2/(10^3)^2}$$

$$|H(\omega = 10^6)| = \frac{\sqrt{1+10^6}}{1+10^6}$$

Approximately:

$$|H(\omega = 10^6)| \approx \frac{\sqrt{10^6}}{10^6} = \frac{10^3}{10^6} = 10^{-3}$$

$$|H(\omega = 10^6)| \approx 10^{-3}$$

d) **Write an exact expression for the *phase* of $H(j\omega = j1)$, and give an approximate numerical answer.**

Solution

We obtained this expression for the transfer function's phase above:

$$\angle H(\omega) = \text{atan2}(-\frac{\omega}{\omega_c}, 1)$$

Plugging in for $\omega = 1$:

$$\angle H(\omega = 1) = \text{atan2}(-\frac{10^0}{10^3}, 1) = \tan^{-1}(-10^{-3})$$

By the small angle approximation, this is:

$$\angle H(\omega = 1) \approx -10^{-3}\text{rad}$$

e) **Write down an expression for the time-domain output waveform $V_{out}(t)$ of this filter if the input voltage is $V(t) = 1\sin(1000t)$ V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

Solution

We can find the transfer function at this point:

$$|H(\omega = 10^3)| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

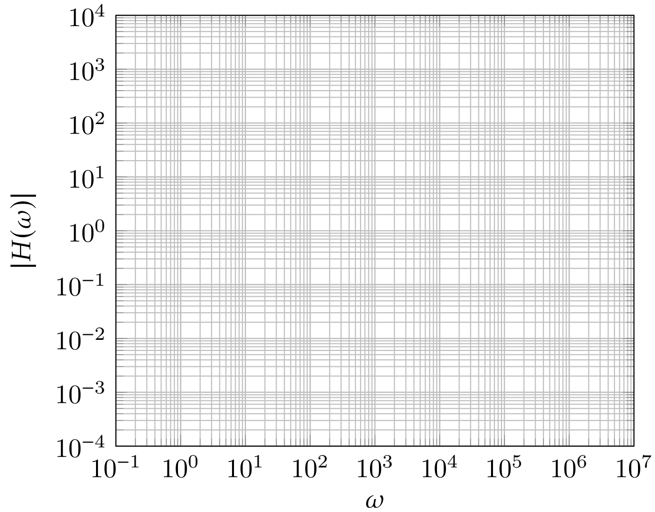
$$\angle H(\omega = 10^3) = \text{atan2}(-1, 1) = -45°$$

Therefore the output will be:

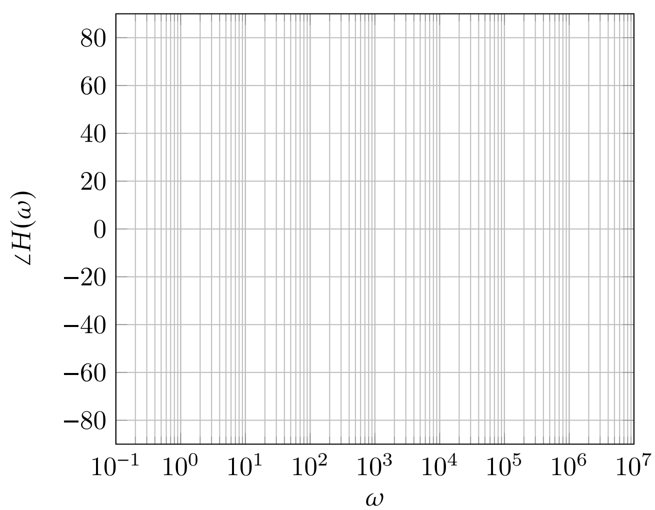
$$V_{out}(t) = \frac{1}{\sqrt{2}} \sin\left(1000t - 45°\right)$$

f) **Use a computer or calculator to help you sketch the Bode plot (both magnitude and phase) of the filter on the graph paper below.**

Log-log plot of transfer function magnitude

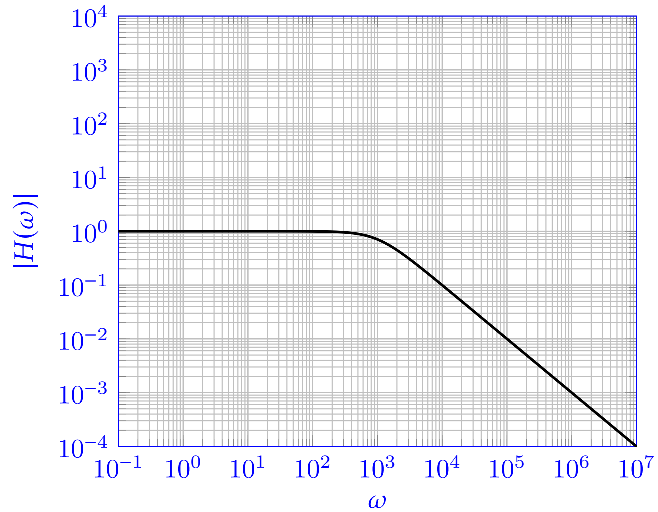


Semi-log plot of transfer function phase

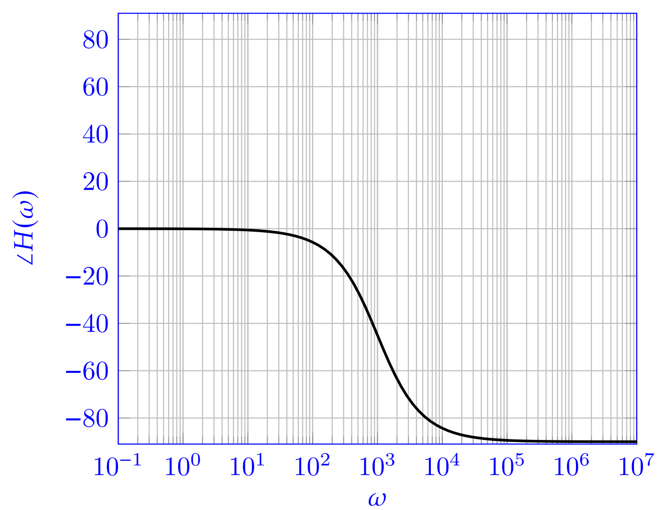


Solution

Log-log plot of transfer function magnitude



Semi-log plot of transfer function phase



3 Color Organ Filter Design

In the fourth lab, we will design low-pass, band-pass, and high-pass filters for a color organ. There are red, green, and blue LEDs. Each color will correspond to a specified frequency range of the input audio signal. The intensity of the light emitted will correspond to the amplitude of the audio signal.

a) First, you realize that you can build simple filters using a resistor and a capacitor. Design the first-order **passive** low and high pass filters with following frequency ranges for each filter using $1\,\mu\text{F}$ capacitors. ("Passive" means that the filter does not require any power supply.)

- Low pass filter – 3-dB frequency at $2400\,\text{Hz} = 2\pi \cdot 2400\frac{rad}{sec}$
- High pass filter – 3-dB frequency at $100\,\text{Hz} = 2\pi \cdot 100\frac{rad}{sec}$

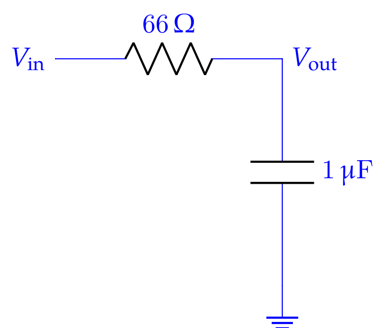
Draw the schematic-level representation of your designs and show your work finding the resistor values. Also, please mark V_{in}, V_{out}, and ground nodes in your schematic. Round your results to two significant figures.

Solution

a) Low-pass filter

$$f_{3\,\text{dB}} = \frac{1}{2\pi RC} = 2400\,\text{Hz}$$

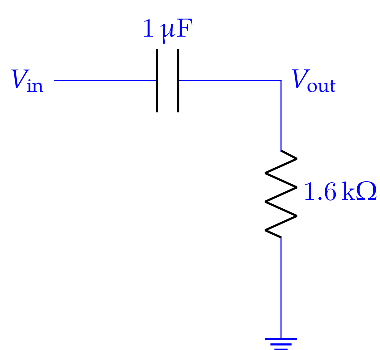
Therefore, we need a $66\,\Omega$ resistor.



b) High-pass filter

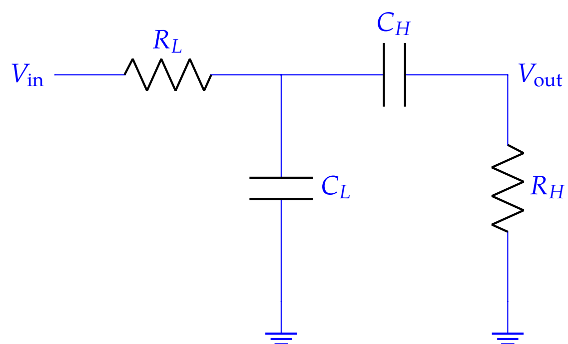
$$f_{3\,\text{dB}} = \frac{1}{2\pi RC} = 100\,\text{Hz}$$

Therefore, we need a $1.6\,\text{k}\Omega$ resistor.



b) You decide to build a bandpass filter by simply cascading the first-order low-pass and high-pass filters you designed in part (a). Connect the V_{out} node of your low-pass filter directly to the V_{in} node of your high pass filter. The V_{in} of your new band-pass filter is the V_{in} of your old low-pass filter, and the V_{out} of the new filter is the V_{out} of your old high-pass filter. What is H_{BPF}, the transfer function of your new band-pass filter? Use R_L, C_L, R_H, and C_H for low-pass filter and high-pass filter components, respectively. Show your work.

Solution



$$\left(\frac{1}{j\omega C_H} + R_H\right) \parallel \frac{1}{j\omega C_L} = \frac{\left(\frac{1}{j\omega C_H} + R_H\right)\frac{1}{j\omega C_L}}{\frac{1}{j\omega C_L} + \frac{1}{j\omega C_H} + R_H} = \frac{1 + j\omega R_H C_H}{-\omega^2 R_H C_L C_H + j\omega(C_H + C_L)}$$

Therefore, the transfer function from V_{in} of **the low pass filter** to V_{out} of **the low pass filter** is

$$H_{LPF} = \frac{\left(\frac{1}{j\omega C_H} + R_H\right) \parallel \frac{1}{j\omega C_L}}{R_L + \left(\frac{1}{j\omega C_H} + R_H\right) \parallel \frac{1}{j\omega C_L}} = \frac{1 + j\omega R_H C_H}{1 - \omega^2 R_L R_H C_L C_H + j\omega(R_H C_H + R_L C_L + R_L C_H)}$$

And, the transfer function from V_{out} of **the low pass filter** to V_{out} of **the high pass filter** is

$$H_{HPF} = \frac{j\omega R_H C_H}{1 + j\omega R_H C_H}$$

The overall transfer function is

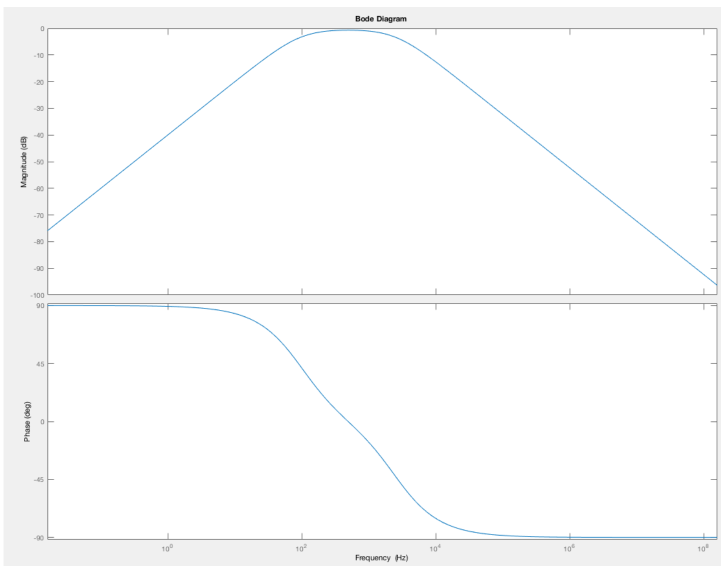
$$H_{BPF} = H_{LPF} \cdot H_{HPF} = \frac{j\omega R_H C_H}{1 - \omega^2 R_L R_H C_L C_H + j\omega(R_H C_H + R_L C_L + R_L C_H)}$$

c) Plug the component values you found in (a) into the transfer function H_{BPF}. Using MATLAB or IPython, draw a Bode plot from 0.1 Hz to 1 GHz. If you use iPython, you may find the function `scipy.signal.bode` useful. What are the frequencies at which the numerator and denumerator of the transfer function become zero? What is the maximum magnitude of H_{BPF} in dB? Is that something that you want? If not, explain why not and suggest a simple way (either adding passive or active components) to fix it.

Solution

$$H_{BPF} = \frac{j\omega(1.6 \cdot 10^{-3})}{1 - \omega^2(1.1 \cdot 10^{-7}) + j\omega(1.7 \cdot 10^{-3})}$$

The Bode plot is as below.



There are two roots for denumerator and one root for numerator at 100 Hz, 2.4 kHz, and DC, respectively. The maximum magnitude (around 500 Hz $= 3.14 \times 10^3 \ \frac{\text{rad}}{\text{s}}$) is

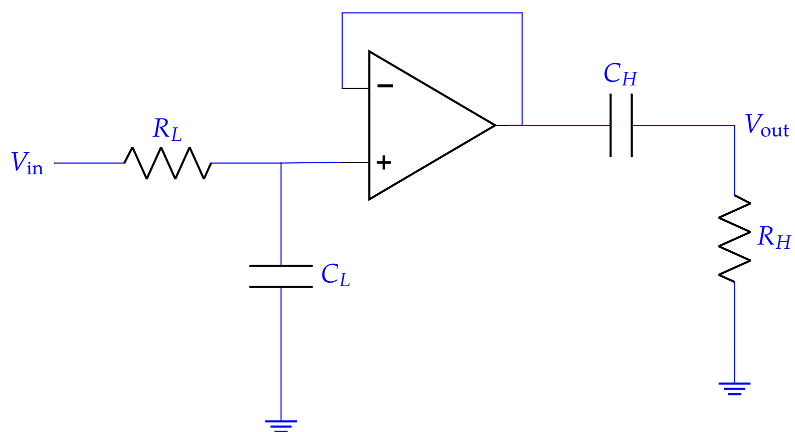
$$\left| \frac{j(3.14 \cdot 10^3)(1.6 \cdot 10^{-3})}{1 - (3.14 \cdot 10^3)^2(1.1 \cdot 10^{-7}) + j(3.14 \cdot 10^3)(1.7 \cdot 10^{-3})} \right| = 0.94 \ \frac{\text{V}}{\text{V}} = -0.52 \, \text{dB}$$

This is pretty similar to what we wanted. The gain, $|H_{BPF}|$, is close to 0 dB at its maximum. However, the transfer function of the bandpass filter that we likely intended to get by cascading the two filter circuits was:

$$
\begin{aligned}
H_{\text{ideal BPF}} &= \frac{j\omega R_H C_H}{(1 + j\omega R_H C_H)(1 + j\omega R_L C_L)} \\
&= \frac{j\omega R_H C_H}{1 - \omega^2 R_H C_H R_L C_L + j\omega(R_L C_L + R_H C_H)}
\end{aligned}
$$

Therefore, in our circuit, only the $j\omega R_L C_H$ term is added at the denominator. Because $R_L = 66\,\Omega$ is small, it did not cause any significant problem in our case. $j\omega R_L C_H$ is added because the low pass filter is experiencing impedance loading from the high pass filter, leading to a change in H_{LPF}. However, to be safe, a simple solution is to place a voltage buffer between the filters as below.

Note that the ideal voltage buffer has infinite input impedance and zero output impedance. This blocks any load effects from the following stage, and the next stage will see the op-amp output as an ideal voltage source.



d) Now that you know how to make filters and amplifiers, we can finally build a system for the color organ circuit below. Before going into the actual schematic design, you must first set specifications for each block. The goal of the circuit is to divide the input signal into three frequency bands and turn the LEDs on based on the input signal's frequency.

In this problem, assume that the mic board's transfer function is of the following form:

$$
V_{MIC} = K_{MIC} \frac{j\omega \left(1 + \frac{j\omega}{\omega_{z1}}\right)}{\left(1 + \frac{j\omega}{\omega_{p1}}\right)\left(1 + \frac{j\omega}{\omega_{p2}}\right)\left(1 + \frac{j\omega}{\omega_{p3}}\right)}
$$

where K_{MIC} is a constant gain, $\omega_{z1} = 2\pi.200Hz$, $\omega_{p1} = 2\pi.10Hz$, $\omega_{p2} = 2\pi.100Hz$, and $\omega_{p3} = 2\pi.10KHz$. The magnitude of the voltage at the mic board output is $1\,\text{V}$ peak-to-peak at $40\,\text{Hz}$. (*Hint:* You can use this information to calculate K_{MIC}.)

Suppose that the three filters have transfer functions as below.

- Low pass filter

$$H_{LPF} = \frac{2}{1 + \frac{j\omega}{200\pi}}$$

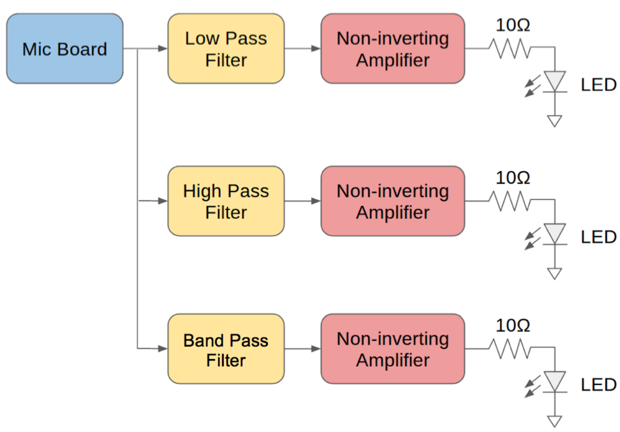
- Band pass filter

$$H_{BPF} = \frac{4.54 \cdot 10^{-4} j\omega}{\left(1 + \frac{j\omega}{400\pi}\right)\left(1 + \frac{j\omega}{4000\pi}\right)}$$

- High pass filter

$$H_{HPF} = \frac{\frac{j\omega}{8000\pi}}{1 + \frac{j\omega}{8000\pi}}$$

What are the phasor voltages at the output of each filter as a function of ω? To clarify, $\frac{3(1+j\omega(1.5\cdot10^3))}{1+j\omega(2\cdot100)}$ would be a valid phasor voltage at the output of some filter. Assume that there are ideal voltage buffers before and after each filter.



Solution

Because we know that we have $1\ V_{pp}$ at $40\,\text{Hz}$, we can plug $2\pi \cdot 40$ into ω to get K_{MIC}.

$$1 = \left| K \cdot \frac{j(80\pi)\left(1 + \frac{j(80\pi)}{\omega_{z1}}\right)}{\left(1 + \frac{j(80\pi)}{\omega_{p1}}\right)\left(1 + \frac{j(80\pi)}{\omega_{p2}}\right)\left(1 + \frac{j(80\pi)}{\omega_{p3}}\right)} \right|$$

Therefore, $K = 0.017$. Finally, the phasor voltages at the output of each filter are as below.

$$V_{LPF} = 0.034 \cdot \frac{j\omega\left(1 + \frac{j\omega}{400\pi}\right)}{\left(1 + \frac{j\omega}{20\pi}\right)\left(1 + \frac{j\omega}{200\pi}\right)^2\left(1 + \frac{j\omega}{20000\pi}\right)}$$

$$V_{BPF} = 7.72 \cdot 10^{-6} \cdot \frac{(j\omega)^2}{\left(1 + \frac{j\omega}{20\pi}\right)\left(1 + \frac{j\omega}{200\pi}\right)\left(1 + \frac{j\omega}{4000\pi}\right)\left(1 + \frac{j\omega}{20000\pi}\right)}$$

$$V_{HPF} = 0.017 \cdot \frac{\frac{(j\omega)^2}{8000\pi}\left(1 + \frac{j\omega}{400\pi}\right)}{\left(1 + \frac{j\omega}{20\pi}\right)\left(1 + \frac{j\omega}{200\pi}\right)\left(1 + \frac{j\omega}{8000\pi}\right)\left(1 + \frac{jw}{20000\pi}\right)}$$

e) For $50\,\text{Hz}$, $1000\,\text{Hz}$, and $8000\,\text{Hz}$, what is the voltage gain required of each non-inverting amplifier such that the output peak to peak voltage measured right before the $10\,\Omega$ resistor is 5 V_{pp}?

Solution

a) Low pass filter path
At $\omega = 100\pi$,

$$|V_{LPF}| = \left| 0.034 \cdot \frac{j100\pi\left(1 + \frac{j100\pi}{400\pi}\right)}{\left(1 + \frac{j100\pi}{20\pi}\right)\left(1 + \frac{j100\pi}{200\pi}\right)^2\left(1 + \frac{j100\pi}{20000\pi}\right)} \right| = 1.73$$

Therefore, the non-inverting amplifier gain should be $2.9\,\frac{\text{V}}{\text{V}}$ (or $9.24\,\text{dB}$).

b) Band pass filter path
At $\omega = 2000\pi$,

$$|V_{BPF}| = \left| \frac{7.72 \cdot 10^{-6} \cdot (j2000\pi)^2}{\left(1 + \frac{j2000\pi}{20\pi}\right)\left(1 + \frac{j2000\pi}{200\pi}\right)\left(1 + \frac{j2000\pi}{4000\pi}\right)\left(1 + \frac{j2000\pi}{20000\pi}\right)} \right| = 0.27$$

Therefore, the non-inverting amplifier gain should be $18.5\,\frac{\text{V}}{\text{V}}$ (or $25.3\,\text{dB}$).

c) High pass filter path

At $\omega = 16000\pi$,

$$|V_{HPF}| = \left| \frac{0.017 \cdot \frac{(j16000\pi)^2}{8000\pi} \left(1 + \frac{j16000\pi}{400\pi}\right)}{\left(1 + \frac{j16000\pi}{20\pi}\right)\left(1 + \frac{j16000\pi}{200\pi}\right)\left(1 + \frac{j16000\pi}{8000\pi}\right)\left(1 + \frac{j16000\pi}{20000\pi}\right)} \right| = 0.37$$

Therefore, the non-inverting amplifier gain should be $13.5\,\frac{\text{V}}{\text{V}}$ (or $22.6\,\text{dB}$).

4 Mystery Microphone

You are working for Mysterious Miniature Microphone Multinational when your manager asks you to test a batch of the company's new microphones. You grab one of the new microphones off the shelf, use a tone generator [1] to play pure tones of uniform amplitude at various frequencies, and measure the resultant peak-to-peak voltages using an oscilloscope. You collect data, and then plot it (on a logarithmic scale). The plot is shown below:

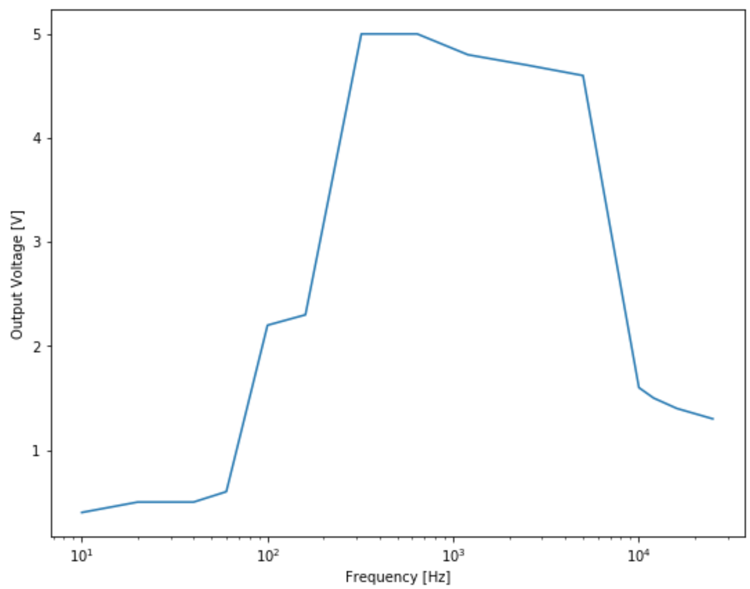


Figure 8: Frequency Response

a) To which frequencies is the microphone most sensitive, and to which frequencies is the microphone least sensitive?

Solution

The microphone is most sensitive to frequencies in the range of 320 Hz to 5 kHz, and least sensitive below $\approx$ 100 Hz or so.

You report these findings to your manager, who thanks you for the preliminary data and proceeds to co-ordinate some human listener tests.

[1]Note that soundwaves are simply sinusoids at various frequencies with some amplitude and phase. The microphone's diaphragm oscillates with the sound (pressure) waves, moving the attached wire coil back and forth over an internal magnet, which induces a current in the wire. In this way, a microphone can be modeled as a signal-dependent current source. The output current can be converted to a voltage by simply adding a known resistor to the circuit and measuring the voltage across that resistor.

In the meantime, your manager asks you to predict the effects of the microphone recordings on human listeners, and encourages you to start thinking more deeply about the relationships.

b) For testing purposes, you have a song with sub-bass ($150\,\text{Hz}$ or less), mid-range ($\approx 1\,\text{kHz}$), and some high frequency electronic parts ($> 12\,\text{kHz}$). Which frequency ranges of the song would you be able to hear easily, and which parts would you have trouble hearing? Why?

Solution

The mid-range would be most audible since the amplitude is the highest at these frequencies. The high frequency electronic parts are the next loudest. The sub-bass parts would be the parts you have trouble hearing since the output amplitude is so low.

c) After a few weeks, your manager reports back to you on the findings. Apparently, this microphone causes some people's voices to sound really weird, resulting in users threatening to switch to products from a competing microphone company.

It turns out that we can design some filters to "fix" the frequency response so that the different frequencies can be recorded more equally, thus avoiding distortion. Imagine that you have a few (say up to 4 or so) blocks. Each of these blocks detects a set range of frequencies, and if the signal is within this range, it will switch on a op-amp circuit of your choice. For example, it can be configured to switch on an op-amp filter to double the voltage for signals between $100\,\text{Hz}$ and $200\,\text{Hz}$.

What ranges of signals would require such a block, and what gain would you apply to each block such that the resulting peak-to-peak voltage is about $5\,\text{V}$ for all frequencies?

Solution

The output amplitude for $< 100\,\text{Hz}$ is $\approx 0.5\,\text{V}$, so it needs a gain of 10.

For 100-160Hz, the amplitude is $\approx 2.5\,\text{V}$, so it needs a gain of 2.

320-5000Hz already has an amplitude of 5V, so no gain is needed.

10000-20000Hz has an amplitude of $\approx 1.5\,\text{V}$, so it needs a gain of 3.33.

5 (OPTIONAL) Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very effective way to really learn material. Having some practice at trying to create problems helps you study for exams much better than simply solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really consolidate your understanding of the course material.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Roughly how many total hours did you work on this homework?**