EE16B - Spring'20 - Lecture 6A Notes[1]

Murat Arcak

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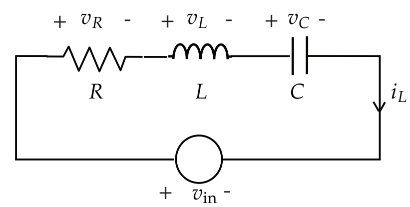
State Space Representation of Dynamical Systems

State variables are internal variables that fully represent the state of a dynamical system at a given time. In previous lectures we used capacitor voltages and inductor currents as state variables of a circuit, and wrote differential equations that tell us how these variables evolve over time. The vector of such variables is called a *state vector* and the vector differential equation governing their evolution is called a *state model*.



Example 1: As a familiar example consider the RLC circuit depicted on the right where v_{in} denotes the input voltage.

Since the capacitor and inductor satisfy the relations

$$C\frac{dv_C(t)}{dt} = i_C(t) \tag{1}$$

$$L\frac{di_L(t)}{dt} = v_L(t), \tag{2}$$

we select v_C and i_L as the state variables. We then eliminate i_C from (1) by noting that $i_C = i_L$, and eliminate v_L from (2) using KVL ($v_L + v_C + v_R = v_{\text{in}}$), Ohm's Law ($v_R = Ri_R$), and $i_R = i_L$:

$$v_L = -v_C - v_R + v_{\text{in}} = -v_C - Ri_L + v_{\text{in}}. \tag{3}$$

Then the state model becomes

$$\begin{aligned} \frac{d}{dt}v_C(t) &= \frac{1}{C}i_L(t) \\ \frac{d}{dt}i_L(t) &= \frac{1}{L}\left(-v_C(t) - Ri_L(t) + v_{\text{in}}(t)\right). \end{aligned} \tag{4}$$

In a state model the left-hand side consists of derivatives of the state variables and the right-hand side depends only on the state variables and external inputs (v_{in} in this example). Other variables appearing in the equations, such as v_L in (2), must be eliminated by expressing them in terms of the state and input variables, as we did in (3).

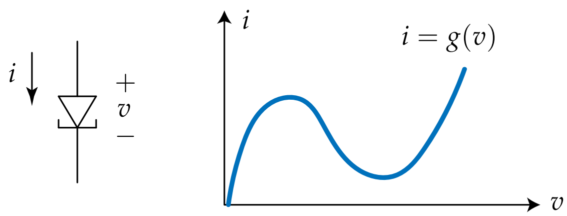
We say that a state model is *linear* if the right-hand side depends linearly on the state and input variables, as in (4) above. For a linear model the right-hand side can be written as a matrix multiplying the state vector, plus another matrix multiplying the input. Thus, for (4),

$$\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_{\text{in}}(t). \tag{5}$$

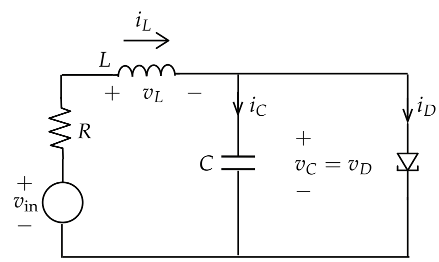
Most physical systems, however, are nonlinear. We have already seen nonlinear voltage-current curves for transistors[2]. The next example studies another nonlinear circuit element, the tunnel diode.

[2] However, since we focused on the low-voltage region where a linear approximation was adequate, we used linear differential equations.

Example 2: A tunnel diode is characterized by a voltage-current curve where, for a certain voltage range, the current decreases with increasing voltage. This is due to a quantum mechanical effect called tunneling.



Now consider the circuit on the right. We again use the state variables i_L and v_C, and start building a state model using the relations



$$C\frac{dv_C(t)}{dt} = i_C(t) \quad (6)$$
$$L\frac{di_L(t)}{dt} = v_L(t). \quad (7)$$

The next task is to rewrite the right-hand side in terms of state variables i_L and v_C, and input v_{in}. To do so note from KCL that $i_C = i_L - i_D$ and subtitute $i_D = g(v_D) = g(V_C)$, since $v_D = v_C$. Thus, (6) becomes

$$C\frac{dv_C(t)}{dt} = i_L(t) - g(v_C(t)), \quad (8)$$

where only the state variables i_L and v_C appear on the right-hand side. Likewise, using KVL, we substitute $v_L = -v_C - Ri_L + v_{\text{in}}$ in (7) and obtain

$$L\frac{di_L(t)}{dt} = -v_C(t) - Ri_L(t) + v_{\text{in}}(t). \quad (9)$$

Dividing both sides of (8) by C and both sides of (9) by L, we obtain the state model:

$$\begin{aligned} \frac{d}{dt}v_C(t) &= \frac{1}{C}i_L(t) - \frac{1}{C}g(v_C(t)) \\ \frac{d}{dt}i_L(t) &= \frac{1}{L}\left(-v_C(t) - Ri_L(t) + v_{\text{in}}(t)\right). \end{aligned} \quad (10)$$

Since g is a nonlinear function, (10) is a nonlinear state model and can't be written in the matrix-vector form (5) we used in Example 1 to represent the linear model (4).

General form of State Equations

A general state model with n states and m inputs has the form

$$\begin{aligned}
\frac{d}{dt}x_1(t) &= f_1(x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t)) \\
\frac{d}{dt}x_2(t) &= f_2(x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t)) \\
&\vdots \\
\frac{d}{dt}x_n(t) &= f_n(x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t)),
\end{aligned} \tag{11}$$

where $f_1,\cdots,f_n$ are functions of the state and input variables.

In Examples 1 and 2 above we had $n = 2$ states $x_1 = v_C$ and $x_2 = i_L$, and a single ($m = 1$) input $u = v_{\text{in}}$. Thus (10) has the form above with

$$f_1(x_1,x_2) = \frac{1}{C}x_2 - \frac{1}{C}g(x_1), \quad f_2(x_1,x_2,u) = \frac{1}{L}\left(-x_1 - Rx_2 + u\right).$$

We will henceforth write (11) compactly as

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t),\vec{u}(t)) \tag{12}$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad f(\vec{x},\vec{u}) = \begin{bmatrix} f_1(\vec{x},\vec{u}) \\ f_2(\vec{x},\vec{u}) \\ \vdots \\ f_n(\vec{x},\vec{u}) \end{bmatrix}.$$

The state model (11) is linear if for each $i = 1,\ldots,n$, the function f_i has the form

$$f_i(x_1,\cdots,x_n,u_1,\cdots,u_m) = a_{i1}x_1 + \cdots + a_{in}x_n + b_{i1}u_1 + \cdots + b_{im}u_m,$$

where $a_{i1},\cdots,a_{in},b_{i1},\cdots,b_{im}$ are coefficients. In this case we can write (12) in the matrix-vector form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t), \tag{13}$$

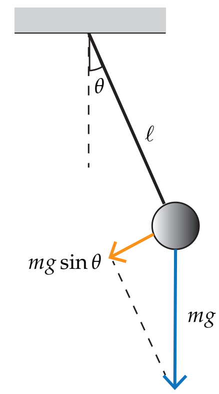
where A is a $n \times n$ matrix and B is a $n \times m$ matrix. The ith column of A consists of the coefficients $a_{i1},\cdots,a_{in}$ and ith column of B consists of $b_{i1},\cdots,b_{im}$. If there is only one input then B is $n \times 1$, that is a column vector, and we may write $\vec{b}$ instead of B:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t).$$

For example, (5) is of this form with

$$A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}.$$

In this module of the course we broaden our scope beyond circuits and analyze other dynamical systems, such as mechanical systems, again using state models. In circuit analysis we selected the state variables to be the inductor currents and capacitor voltages, as these variables are associated with the energy stored in these elements. Likewise, in modeling mechanical systems it is customary to select positions and velocities as the state variables, since the former is associated with potential energy and the latter with kinetic energy.



Example 3: The motion of the pendulum depicted on the right is governed by the differential equation

$$m\ell\frac{d^2\theta(t)}{dt^2} = -k\ell\frac{d\theta(t)}{dt} - mg\sin\theta(t) \tag{14}$$

where the left hand side is mass×accelaration in the tangential direction, and the right hand side is total force acting in that direction, including friction and the tangential component of the gravitational force.

To bring this second order differential equation to state space form we define the state variables to be the angle and angular velocity:

$$x_1(t) := \theta(t) \qquad x_2(t) := \frac{d\theta(t)}{dt},$$

and note that they satisfy

$$\begin{aligned} \frac{d}{dt}x_1(t) &= x_2(t) \\ \frac{d}{dt}x_2(t) &= -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t). \end{aligned} \tag{15}$$

The first equation here follows from the definition of $x_2(t)$ as the angular velocity, and the second equation follows from (14).

Here we did not consider external forces that could act as inputs, so the equations (15) have the form (12) with the input omitted:

$$f(\vec{x}) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix}. \tag{16}$$

Equilibrium States

For a system without inputs, $\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t))$, the solutions of the static equation

$$f(\vec{x}) = 0$$

are called *equilibrium points*. If we pick an equilibrium point $\vec{x}^*$ as the initial state at t_0, then

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}^*) = 0$$

for all $t \geq t_0$, therefore the state remains at $\vec{x}^*$ in the future:

$$\vec{x}(t) = \vec{x}^* \quad t \geq t_0.$$

Example 3 revisited: We can find the equilibrium points for the pendulum example above by solving the equation

$$f(\vec{x}) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix} = 0.$$

This consists of two equations,

$$x_2 = 0, \quad -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 = 0,$$

which have two distinct solutions:

$$x_1 = 0, \quad x_2 = 0,$$

that is the downward pointing position of the pendulum, and

$$x_1 = \pi, \quad x_2 = 0,$$

which is the upright position[3]. As this example illustrates, a system may have more than one equilibrium. We will see later that the upright position is *unstable*, meaning that the pendulum would diverge from this equilibrium when slightly perturbed. In contrast the downward position is *stable*, because the pendulum would return to this position after some oscillations with the help of the friction term.

[3] Other solutions, such as $(x_1, x_2) = (2\pi, 0)$, or $(x_1, x_2) = (3\pi, 0)$ are identical to one of the two equilibria already described.