# *EE16B - Spring'20 - Lecture 7A Notes*[1]

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## *Linearization and Discrete-Time Systems*

### *Linearization with Inputs*

In the last lecture we considered nonlinear systems with no inputs and linearized them by applying a Taylor approximation around an equilibrium. We can also apply linearization to systems with inputs,

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t)),$$

around an equilibrium $\vec{x}^*$ maintained by a constant input $\vec{u}^*$ that satisfies $f(\vec{x}^*, \vec{u}^*) = 0$.

Define the perturbation variables $\tilde{x}(t)$ and $\tilde{u}(t)$ as:

$$\boxed{\tilde{x}(t) := \vec{x}(t) - \vec{x}^*, \quad \tilde{u}(t) := \vec{u}(t) - \vec{u}^*.} \tag{1}$$

Then,

$$\begin{aligned}
\frac{d}{dt}\tilde{x}(t) &= \frac{d}{dt}\vec{x}(t) - \frac{d}{dt}\vec{x}^* \\
&= \frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t)) = f(\vec{x}^* + \tilde{x}(t), \vec{u}^* + \tilde{u}(t)) \\
&\approx f(\vec{x}^*, \vec{u}^*) + \nabla_x f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*}\, \tilde{x}(t) + \nabla_u f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*}\, \tilde{u}(t) \qquad (2)
\end{aligned}$$

where

$$\begin{aligned}
\nabla_x f(\vec{x}, \vec{u}) &:= \begin{bmatrix} \frac{\partial f_1(\vec{x},\vec{u})}{\partial x_1} & \frac{\partial f_1(\vec{x},\vec{u})}{\partial x_2} & \cdots & \frac{\partial f_1((\vec{x},\vec{u})}{\partial x_n} \\ \frac{\partial f_2(\vec{x},\vec{u})}{\partial x_1} & \frac{\partial f_2(\vec{x},\vec{u})}{\partial x_2} & \cdots & \frac{\partial f_2((\vec{x},\vec{u})}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n(\vec{x},\vec{u})}{\partial x_1} & \frac{\partial f_n(\vec{x},\vec{u})}{\partial x_2} & \cdots & \frac{\partial f_n(\vec{x},\vec{u})}{\partial x_n} \end{bmatrix} \\
\nabla_u f(\vec{x}, \vec{u}) &:= \begin{bmatrix} \frac{\partial f_1(\vec{x},\vec{u})}{\partial u_1} & \frac{\partial f_1(\vec{x},\vec{u})}{\partial u_2} & \cdots & \frac{\partial f_1((\vec{x},\vec{u})}{\partial u_m} \\ \frac{\partial f_2(\vec{x},\vec{u})}{\partial u_1} & \frac{\partial f_2(\vec{x},\vec{u})}{\partial u_2} & \cdots & \frac{\partial f_2((\vec{x},\vec{u})}{\partial u_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n(\vec{x},\vec{u})}{\partial u_1} & \frac{\partial f_n(\vec{x},\vec{u})}{\partial u_2} & \cdots & \frac{\partial f_n(\vec{x},\vec{u})}{\partial u_m} \end{bmatrix}.
\end{aligned}$$

Substituting $f(\vec{x}^*, \vec{u}^*) = 0$ in (2) and defining

$$\boxed{A := \nabla_x f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*} \quad B := \nabla_u f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*}} \tag{3}$$

we obtain the linearization:

$$\frac{d}{dt}\tilde{x}(t) \approx A\tilde{x}(t) + B\tilde{u}(t).$$

Example 1: The velocity $v(t)$ of a vehicle is governed by

$$M\frac{d}{dt}v(t) = -\frac{1}{2}\rho ac\, v(t)^2 + \frac{1}{R}u(t) \tag{4}$$

where $u(t)$ is the wheel torque, $M$ is vehicle mass, $\rho$ is air density, $a$ is vehicle area, $c$ is drag coefficient, and $R$ is wheel radius. Note that we can maintain the velocity at a desired value $v^*$ if we apply the torque

$$u^* = \frac{R}{2}\rho ac\, v^{*2},$$

which counterbalances the drag force at that velocity. We rewrite the model (4) as $\frac{d}{dt}v(t) = f(v(t), u(t))$, where

$$f(v,u) = -\frac{1}{2M}\rho ac\, v^2 + \frac{1}{RM}u.$$

Then the linearized dynamics for the perturbation $\tilde{v}(t) = v(t) - v^*$ is

$$\frac{d}{dt}\tilde{v}(t) = \lambda\tilde{v}(t) + b\tilde{u}(t), \tag{5}$$

where $\tilde{u}(t) = u(t) - u^*$,

$$\lambda = \left.\frac{\partial f(v,u)}{\partial v}\right|_{v^*,u^*} = -\frac{1}{M}\rho acv^*, \quad b = \left.\frac{\partial f(v,u)}{\partial u}\right|_{v^*,u^*} = \frac{1}{RM}.$$

Here we used the letters $\lambda$ and $b$ instead of $A$ and $B$ to emphasize that they are scalars. Note that if we apply $u(t) = u^*$, that is $\tilde{u}(t) = 0$, then the solution of the scalar differential equation (5) is

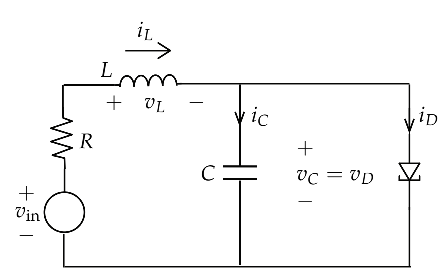
$$\tilde{v}(t) = \tilde{v}(0)e^{\lambda t},$$

which converges to 0 since $\lambda < 0$. This means that if $v(t)$ is perturbed from $v^*$, it will return[2] to $v^*$. Equilibrium points with this property are called *stable*, a concept we will study in detail later.

[2] The rate of convergence depends on $\lambda$. For a typical sedan at $v^* = 29$ m/s ($\approx$ 65 mph) we would get $\lambda \approx -0.01$ sec$^{-1}$ with parameters $M = 1700$ kg, $a = 2.6$ m$^2$, $\rho = 1.2$ kg/m$^3$, $c = 0.2$.

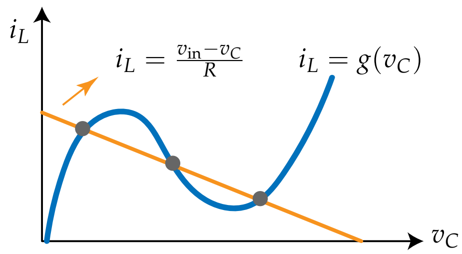
Example 2: In previous lectures we discussed the tunnel diode circuit on the right and obtained the state model:

$$\begin{aligned}\frac{d}{dt}v_C(t) &= \frac{1}{C}i_L(t) - \frac{1}{C}g(v_C(t))\\ \frac{d}{dt}i_L(t) &= \frac{1}{L}\left(-v_C(t) - Ri_L(t) + v_{\text{in}}(t)\right),\end{aligned} \tag{6}$$




where $g$ is a nonlinear function representing the tunnel diode's voltage-current characteristics (see figure below). We also showed that the equilibrium points are the intersections of the curves

$$i_L = g(v_C) \quad \text{and} \quad i_L = \frac{v_{\text{in}} - v_C}{R}. \tag{7}$$




Let $v_{\text{in}}^*$ be a constant input voltage and let $(v_C^*, i_L^*)$ denote one of the resulting equilibrium states, that is one of the intersections of the two curves above. Since the right-hand side of (6) has the form

$$f(v_C, i_L, v_{\text{in}}) = \begin{bmatrix} f_1(v_C, i_L, v_{\text{in}}) \\ f_2(v_C, i_L, v_{\text{in}}) \end{bmatrix} = \begin{bmatrix} \frac{1}{C} i_L - \frac{1}{C} g(v_C) \\ \frac{1}{L}(-v_C - Ri_L + v_{\text{in}}) \end{bmatrix},$$

the matrices $A$ and $B$ in (3) are:

$$A = \begin{bmatrix} \frac{\partial f_1(v_C, i_L, v_{\text{in}})}{\partial v_C} & \frac{\partial f_1(v_C, i_L, v_{\text{in}})}{\partial i_L} \\ \frac{\partial f_2(v_C, i_L, v_{\text{in}})}{\partial v_C} & \frac{\partial f_2(v_C, i_L, v_{\text{in}})}{\partial i_L} \end{bmatrix} \Bigg|_{(v_C^*, i_L^*)} = \begin{bmatrix} \frac{-1}{C} g'(v_C^*) & \frac{1}{C} \\ \frac{-1}{L} & \frac{-R}{L} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1(v_C, i_L, v_{\text{in}})}{\partial v_{\text{in}}} \\ \frac{\partial f_2(v_C, i_L, v_{\text{in}})}{\partial v_{\text{in}}} \end{bmatrix} \Bigg|_{(v_C^*, i_L^*)} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}.$$

## *Discrete-Time Systems*

In a *discrete*-time system, the state vector $\vec{x}(t)$ evolves according to a *difference* equation rather than a differential equation:

$$\vec{x}(t+1) = f(\vec{x}(t), \vec{u}(t)) \quad t = 0, 1, 2, \ldots \tag{8}$$

Here $f(\vec{x}, \vec{u})$ is a function that gives the state vector at the next time instant based on the present values of the states and inputs.

As in the continuous-time case, when $f(\vec{x}, \vec{u}) \in \mathbb{R}^n$ is linear in $\vec{x} \in \mathbb{R}^n$ and $\vec{u} \in \mathbb{R}^m$, we can rewrite it in the form

$$f(\vec{x}, \vec{u}) = A\vec{x} + B\vec{u}$$

where $A$ is $n \times n$ and $B$ is $n \times m$. The state model is then

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t). \tag{9}$$

Example 3: Let $s(t)$ denote the inventory of a manufacturer at the start of the $t$-th business day. The inventory at the start of the next day, $s(t+1)$, is the sum of $s(t)$ and the goods $g(t)$ manufactured, minus the goods $u_1(t)$ sold on day $t$. Assuming it takes a day to do the manufacturing, the amount of goods $g(t)$ manufactured is equal to the raw material available the previous day, $r(t-1)$. The raw material $r(t)$ is equal to the order placed the previous day, $u_2(t-1)$, assuming it takes a day for the order to arrive.

The state variables $s(t)$, $g(t)$, $r(t)$, thus evolve according to the model

$$\begin{aligned} s(t+1) &= s(t) + g(t) - u_1(t) \\ g(t+1) &= r(t) \\ r(t+1) &= u_2(t), \end{aligned} \tag{10}$$

where $u_1$ and $u_2$ are two distinct inputs, one representing the customer demand and the other the manufacturer's raw material order.

Note that this system is linear, and we can write (10) as:

$$\underbrace{\begin{bmatrix} s(t+1) \\ g(t+1) \\ r(t+1) \end{bmatrix}}_{\vec{x}(t+1)} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s(t) \\ g(t) \\ r(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\vec{u}(t)}.$$

Example 4: Let $p(t)$ be the number of EECS professors in a country in year $t$, and let $r(t)$ be the number of industry researchers with a PhD degree. A fraction, $\gamma$, of the PhDs become professors themselves and the rest become industry researchers. A fraction, $\delta$, in each profession leaves the field every year due to retirement or other reasons.

Each professor graduates, on average, $u(t)$ PhD students per year. We treat this number as a control input because it can be manipulated by the government using research funding. This means there will be $p(t)u(t)$ new PhDs in year $t$, and $\gamma p(t)u(t)$ new professors. The state model is then

$$\begin{aligned} p(t+1) &= (1-\delta)p(t) + \gamma p(t)u(t) \\ r(t+1) &= (1-\delta)r(t) + (1-\gamma)p(t)u(t). \end{aligned} \tag{11}$$

Note that this system is nonlinear due to the product of the state variable $p$ with the input $u$. □

When the input $\vec{u}(t)$ in (8) is a constant vector $\vec{u}^*$, the equilibrium points are obtained by solving for $\vec{x}$ in the equation[3]:

$$\vec{x} = f(\vec{x}, \vec{u}^*). \tag{12}$$

[3] Note that the equilibrium condition (12) in discrete time differs from the continuous time condition $0 = f(\vec{x}, \vec{u}^*)$.

If $\vec{x}^*$ satisfies this equation and we start with the initial condition $\vec{x}^*$, the next state is $f(\vec{x}^*, \vec{u}^*)$, which is again $\vec{x}^*$. The same argument applies to subsequent time instants, so $\vec{x}(t)$ remains at $\vec{x}^*$.

For the linear system (9) the equilibrium condition (12) becomes:

$$\vec{x} = A\vec{x} + B\vec{u}^*, \quad \text{or, equivalently} \quad (I-A)\vec{x} = B\vec{u}^*.$$