EE16B - Spring'20 - Lecture 7B Notes[1]

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Discrete-Time Systems and Discretization

Recall that in a *discrete*-time system, the state vector $\vec{x}(t)$ evolves according to a *difference* equation rather than a differential equation:

$$\vec{x}(t+1) = f(\vec{x}(t), \vec{u}(t)) \quad t = 0, 1, 2, \ldots \tag{1}$$

Here $f(\vec{x}, \vec{u})$ is a function that gives the state vector at the next time instant based on the present values of the states and inputs.

As in the continuous-time case, when $f(\vec{x}, \vec{u}) \in \mathbb{R}^n$ is linear in $\vec{x} \in \mathbb{R}^n$ and $\vec{u} \in \mathbb{R}^m$, we can rewrite it in the form

$$f(\vec{x}, \vec{u}) = A\vec{x} + B\vec{u}$$

where A is $n \times n$ and B is $n \times m$. The state model is then

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t). \tag{2}$$

When the input $\vec{u}(t)$ in (1) is a constant vector $\vec{u}^*$, the equilibrium points are obtained by solving for $\vec{x}$ in the equation[2]:

$$\vec{x} = f(\vec{x}, \vec{u}^*). \tag{3}$$

[2] Note that the equilibrium condition (3) in discrete time differs from the continuous time condition $0 = f(\vec{x}, \vec{u}^*)$.

If $\vec{x}^*$ satisfies this equation and we start with the initial condition $\vec{x}^*$, the next state is $f(\vec{x}^*, \vec{u}^*)$, which is again $\vec{x}^*$. The same argument applies to subsequent time instants, so $\vec{x}(t)$ remains at $\vec{x}^*$.

For the linear system (2) the equilibrium condition (3) becomes:

$$\vec{x} = A\vec{x} + B\vec{u}^*, \quad \text{or, equivalently} \quad (I - A)\vec{x} = B\vec{u}^*.$$

Linearization for nonlinear discrete-time systems is performed similarly to continuous-time. The perturbation variables $\tilde{x}(t) := \vec{x}(t) - \vec{x}^*$ and $\tilde{u}(t) := \vec{u}(t) - \vec{u}^*$ satisfy:

$$\begin{aligned} \tilde{x}(t+1) = & \; \vec{x}(t+1) - \vec{x}^* = f(\vec{x}(t), \vec{u}(t)) - \vec{x}^* \\ \approx & \; f(\vec{x}^*, \vec{u}^*) + \nabla_x f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*} \tilde{x}(t) + \nabla_u f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*} \tilde{u}(t) - \vec{x}^*. \end{aligned}$$

Substituting $f(\vec{x}^*, \vec{u}^*) - \vec{x}^* = 0$, which follows because $\vec{x}^*$ is an equilibrium, we get

$$\boxed{\tilde{x}(t+1) \approx A\tilde{x}(t) + B\tilde{u}(t)}$$

where $A = \nabla_x f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*}$ and $B = \nabla_u f(\vec{x}, \vec{u})|_{\vec{x}^*, \vec{u}^*}$.

Changing State Variables

Given the state vector $\vec{x} \in \mathbb{R}^n$ any transformation of the form

$$\vec{z} := T\vec{x}, \tag{4}$$

where T is a $n \times n$ invertible matrix, defines new variables z_i, $i = 1, \ldots, n$, as a linear combination of the original variables $x_1, \ldots, x_n$.

To see how this change of variables affects the state equation

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t),$$

note that

$$\vec{z}(t+1) = T\vec{x}(t+1) = TA\vec{x}(t) + TB\vec{u}(t)$$

and substitute $\vec{x} = T^{-1}\vec{z}$ in the right hand side to obtain:

$$\vec{z}(t+1) = TAT^{-1}\vec{z}(t) + TB\vec{u}(t).$$

Thus the original A and B matrices are replaced with:

$$\boxed{A_{\text{new}} = TAT^{-1}, \quad B_{\text{new}} = TB.} \tag{5}$$

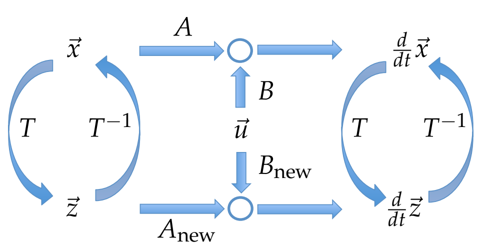
The same change of variables brings the *continuous-time* system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

to the form

$$\frac{d}{dt}\vec{z}(t) = A_{\text{new}}\vec{z}(t) + B_{\text{new}}\vec{u}(t)$$

as depicted below.



We use particular choices of T to obtain special forms of A_{new} and B_{new} that make the analysis easier. For example, we saw in Lecture 3A that we can make A_{new} diagonal if the $n \times n$ matrix A has n independent eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$. This is because the matrix $V = [\vec{v}_1 \cdots \vec{v}_n]$ satisfies

$$AV = [A\vec{v}_1 \cdots A\vec{v}_n] = [\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n] = \underbrace{[\vec{v}_1 \cdots \vec{v}_n]}_{= V} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{=: \Lambda},$$

therefore $V^{-1}AV = \Lambda$. This means that the choice

$$T = V^{-1}$$

gives $A_{\text{new}} = TAT^{-1} = \Lambda$, which is diagonal.

Digital Control

In upcoming lectures we will be designing the input signal $\vec{u}$ of a continuous-time system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{6}$$

to ensure that the solution $\vec{x}(t)$ meets requirements, such as reaching a target state in a given amount of time.

The input signal is typically generated digitally in a computer, by using measurements of $\vec{x}(t)$ sampled every T units of time. Thus the computer receives a discrete sequence

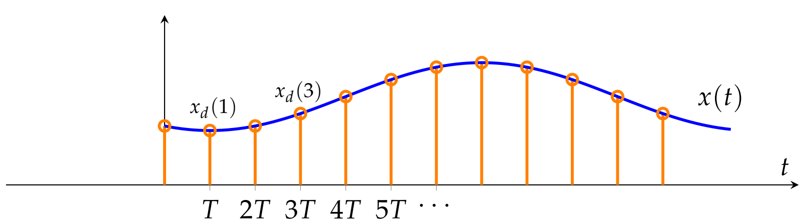
$$\vec{x}(0), \vec{x}(T), \vec{x}(2T), \cdots$$

as shown in the figure below. We use the notation

$$\vec{x}_d(k) := \vec{x}(kT) \tag{7}$$

where the subscript 'd' stands for 'discrete', so that we can represent the samples $\vec{x}(0), \vec{x}(T), \vec{x}(2T), \cdots$ as a discrete-time signal

$$\vec{x}_d(0), \vec{x}_d(1), \vec{x}_d(2), \cdots$$



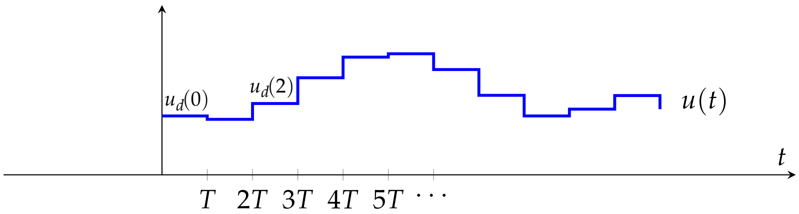
Using this sequence an appropriate control algorithm generates inputs to the system, again as a discrete sequence

$$\vec{u}_d(0), \vec{u}_d(1), \vec{u}_d(2), \ldots$$

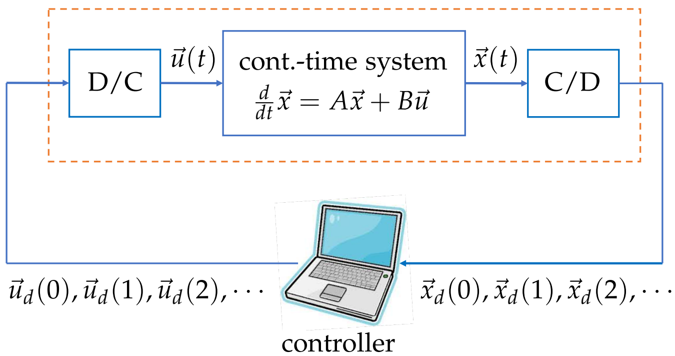
However, since the system (6) admits only continuous-time inputs, this sequence must be converted to continuous-time. This is typically done with a *zero-order hold* device that keeps $\vec{u}(t)$ constant at $\vec{u}_d(0)$ in the interval $t \in [0, T)$, at $\vec{u}_d(1)$ for $t \in [T, 2T)$, and so on. Therefore,

$$\vec{u}(t) = \vec{u}_d(k) \quad t \in [kT, (k+1)T), \tag{8}$$

which has a staircase shape as shown below.



The overall control scheme is illustrated below where the D/C (discrete-to-control) block represents zero-order hold and the C/D (continuous-to-discrete) block represents sampling.



Discretization

From the viewpoint of the controller, the system combined with D/C and C/D blocks (dashed box in the figure above) receives a discrete input sequence $\vec{u}_d(k)$ and generates a discrete state sequence $\vec{x}_d(k)$ that consists of snapshots of $\vec{x}(t)$.

We now wish to derive a discrete-time model

$$\vec{x}_d(k+1) = A_d\vec{x}_d(k) + B_d\vec{u}_d(k) \tag{9}$$

that describes how the state evolves from one snapshot to the next. That is, we want (9) to return the next sample of the continuous-time system (6) when the input $\vec{u}(t)$ is constant in between the samples.

To see how such a discrete-time model can be derived, first assume the continuous-time system has a single state and single input:

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t). \tag{10}$$

Since the value of $x(t)$ at $t = kT$ is $x_d(k)$, the solution of the scalar differential equation above with initial time kT is

$$x(t) = e^{\lambda(t-kT)}x_d(k) + \int_{kT}^{t} e^{\lambda(t-\tau)}bu(\tau)d\tau.$$

We also know that the input $u(t)$ from $t = kT$ to $t = kT + T$ is the constant $u_d(k)$. Thus, the solution at time $t = kT + T$ is

$$x(kT+T) = e^{\lambda T} x_d(k) + \int_{kT}^{kT+T} e^{\lambda(kT+T-\tau)} b u_d(k) d\tau.$$

Substituting $x(kT+T) = x_d(k+1)$ and factoring $bu_d(k)$ out of the integral (since it is constant) we get

$$x_d(k+1) = e^{\lambda T} x_d(k) + \left(\int_{kT}^{kT+T} e^{\lambda(kT+T-\tau)} d\tau \right) b u_d(k). \tag{11}$$

We next simplify the integral in brackets by defining the variable $s := kT + T - \tau$:

$$\int_{kT}^{kT+T} e^{\lambda(kT+T-\tau)} d\tau = \int_{T}^{0} e^{\lambda s}(-ds) = \int_{0}^{T} e^{\lambda s} ds.$$

Substituting in (11) we conclude

$$x_d(k+1) = \lambda_d x_d(k) + b_d u_d(k) \tag{12}$$

where

$$\lambda_d = e^{\lambda T}, \quad b_d = b \int_0^T e^{\lambda s} ds = \begin{cases} bT & \text{if } \lambda = 0 \\ b\frac{e^{\lambda T}-1}{\lambda} & \text{if } \lambda \neq 0. \end{cases}$$

Thus, (12) evaluates the state of the continuous-time model (10) at the next sample time. We refer to (12) as the 'discretization' of (10).