EE16B - Spring'20 - Lecture 8B Notes[1]

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Controllability and System Identification

Controllability Continued

Recall from the last lecture that the discrete-time system

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t), \quad \vec{x}(t) \in \mathbb{R}^n \tag{1}$$

is called controllable if, for every $\vec{x}_{\text{target}} \in \mathbb{R}^n$, there exist a t and an input sequence $\vec{u}(0), \vec{u}(1), \ldots, \vec{u}(t-1)$ such that $x(t) = \vec{x}_{\text{target}}$.

To investigate controllability we assumed B is a column vector $\vec{b} \in \mathbb{R}^n$, and wrote the solution of (1) as

$$\vec{x}(t) - A^t\vec{x}(0) = \begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{t-1}\vec{b} \end{bmatrix} \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(0) \end{bmatrix}. \tag{2}$$

Next we observed that the system is controllable if

$$\text{span}\{\vec{b}, A\vec{b}, \cdots, A^{t-1}\vec{b}\} = \mathbb{R}^n \text{ for some } t \tag{3}$$

because, then, we can choose $u(0), \cdots, u(t-1)$ to match the right-hand side of (2) to $\vec{x}_{\text{target}} - A^t\vec{x}(0)$ for any $\vec{x}_{\text{target}} \in \mathbb{R}^n$.

Henceforth we assume $\vec{b} \neq 0$, since it is trivial to conclude uncontrollability otherwise. We also assume $n \geq 2$, since $b \neq 0$ already guarantees controllability when $n = 1$.

Now imagine an algorithm that starts with $t = 1$, checks if (3) holds; if not, increments t by one and checks (3) again, and so on. Two scenarios are possible:

1. The span grows with every increase in t up to and including $t = n$, at which point we have n linearly independent columns:

$$\text{span}\{\vec{b}, A\vec{b}, \cdots, A^{n-2}\vec{b}, A^{n-1}\vec{b}\} = \mathbb{R}^n.$$

Therefore, the system is controllable.

2. The span grows with every increase in t up to and including $t = m$, where $m < n$, and incrementing t to $t = m + 1$ does not grow the span further – the dimension is still m.

In the second scenario we may be tempted to increase t further and expect that the span may eventually start growing again. This, however, is futile and the dimension of the span will be stuck at $m < n$ no matter how much we increase t.

Here is why: since the span stopped growing when t was raised from $t = m$ to $t = m + 1$, this means the new column $A^m\vec{b}$ was a linear combination of the previous columns $\vec{b}, A\vec{b}, \cdots, A^{m-1}\vec{b}$, that is

$$A^m\vec{b} = \alpha_0\vec{b} + \alpha_1 A\vec{b} + \cdots + \alpha_{m-1}A^{m-1}\vec{b} \tag{4}$$

for some coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$. Raising t further to $m+2$ means adding the new column $A^{m+1}\vec{b}$, but

$$A^{m+1}\vec{b} = A(A^m\vec{b}) = \alpha_0 A\vec{b} + \alpha_1 A^2\vec{b} + \cdots + \alpha_{m-1}A^m\vec{b}$$

and substituting (4) for the last term in this sum, we see that $A^{m+1}\vec{b}$ is still a linear combination of $\vec{b}, A\vec{b}, \cdots, A^{m-1}\vec{b}$. The same argument applies to subsequent columns $A^{m+2}\vec{b},\ A^{m+3}\vec{b}, \ldots$, which means that the dimension of the span remains stuck at m.

Therefore, in scenario 2, the span in (3) will not reach $\mathbb{R}^n$ no matter how much we increase t, and the system is uncontrollable.

It follows from the discussion above that, instead of checking (3) for varying values of t, we need to only check it for $t = n$. If it holds for $t = n$, then scenario 1 applies and the systems is controllable. If not, scenario 2 applies and the system is uncontrollable. This leads to the following simplified controllability test:

$$\text{Controllability} \quad \Leftrightarrow \quad \text{span}\{\vec{b}, A\vec{b}, \cdots, A^{n-2}\vec{b}, A^{n-1}\vec{b}\} = \mathbb{R}^n.$$

Extensions to Multi-Input and Continuous-Time Systems

The controllability test above was derived for the single-input case where B is a single column $\vec{b}$. The same test is also applicable to a multi-input system[2] where B is $n \times m$. In this case we form the *controllability matrix*

$$C = \begin{bmatrix} B & AB & \cdots A^{n-1}B \end{bmatrix}$$

which now has nm columns, and check whether its column space is $\mathbb{R}^n$. The system is controllable if so, and uncontrollable otherwise.

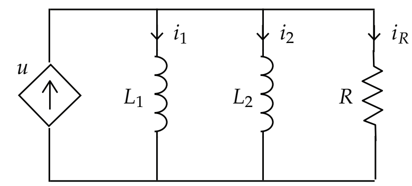
The controllability condition for the continuous-time system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

[2] The derivation for the multi-input case uses the Cayley-Hamilton Theorem that was alluded to in Discussion 8B. This theorem is beyond the scope of this course, but you can consult the Wikipedia article if you are interested.

is exactly the same: form the controllability matrix C above and check whether its column space is $\mathbb{R}^n$. We omit the derivation for this case but illustrate the result with a circuit example.

Example 1: For the circuit depicted on the right we treat the current source as the control $u(t)$, and the inductor currents $i_1(t)$ and $i_2(t)$ as the state variables.



Since the voltage across each capacitor is the same as the voltage across the resistor, we have

$$\begin{aligned} L_1 \frac{di_1(t)}{dt} &= Ri_R(t) \\ L_2 \frac{di_2(t)}{dt} &= Ri_R(t). \end{aligned} \tag{5}$$

Substituting $i_R = u - i_1 - i_2$ from KCL and dividing the equations by L_1 and L_2 respectively, we get

$$\begin{bmatrix} \frac{di_1(t)}{dt} \\ \frac{di_2(t)}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix}}_{A} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix}}_{\vec{b}} u(t).$$

Note that

$$A\vec{b} = \begin{bmatrix} -\frac{R}{L_1}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \\ -\frac{R}{L_2}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \end{bmatrix} = -\left(\frac{R}{L_1} + \frac{R}{L_2}\right)\vec{b}$$

which means that $A\vec{b}$ and $\vec{b}$ are linearly dependent. Thus the system is *not* controllable.

To see the physical obstacle to controllability note that the two inductors in parallel share the same voltage:

$$L_1 \frac{di_1(t)}{dt} = L_2 \frac{di_2(t)}{dt}.$$

Thus,

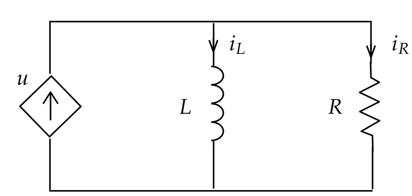
$$\frac{d}{dt}\left(L_1 i_1(t) - L_2 i_2(t)\right) = 0$$

which means that $L_1 i_1(t) - L_2 i_2(t)$ remains constant no matter what u we apply: $L_1 i_1(t) - L_2 i_2(t) = L_1 i_1(0) - L_2 i_2(0)$. Because of this constraint we can't control i_1 and i_2 independently. For example, if $i_1(0) = i_2(0) = 0$, then $L_1 i_1(t) - L_2 i_2(t) = 0$ for all t, and we can't move i_1 and i_2 to target values that don't meet this constraint.

We can, however, control the total current $i_L = i_1 + i_2$ which obeys, from (5),

$$\frac{di_L(t)}{dt} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right) Ri_R(t) = \frac{R}{L}(-i_L(t) + u(t))$$

where $L \triangleq \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$. Note that this is the governing equation for the circuit on the right where the two inductors are lumped into one.



System Identification

In many applications the matrices A and B in the state model

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

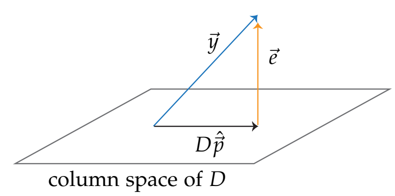
are not known exactly and change with operating conditions. The goal in system identification is to learn these matrices by observing the input sequence and the resulting state sequence.

Before we explain how this is done, let's recall Least Squares estimation from 16A. Suppose we have the relation

$$\vec{y} = D\vec{p} + \vec{e} \tag{6}$$

where $\vec{y} \in \mathbb{R}^{\ell}$ is a vector of measurements, $\vec{p} \in \mathbb{R}^{k}$ is a vector of unknown parameters, D is a known $\ell \times k$ matrix, and $\vec{e}$ represents uncertainty, *e.g.*, due to measurement error. We assume $k < \ell$, which means we have fewer unknowns than measurements.

Least Squares gives an estimate $\hat{\vec{p}}$ such that $D\hat{\vec{p}}$ is as close to $\vec{y}$ as possible, *i.e.*, $\hat{\vec{p}}$ fits the measurements with the least magnitude of error, $\|\vec{e}\|$. As you saw in 16A, this is achieved when $D\hat{\vec{p}}$ matches the projection of $\vec{y}$ onto the column space of D, as depicted on the right.



In this case $\vec{e}$ is orthogonal to the column space of D, which means it is orthogonal to each column of $D = [\vec{d}_1 \cdots \vec{d}_k]$:

$$\vec{d}_i^T \vec{e} = 0, \quad i = 1, 2, \ldots, k, \quad \text{or equivalently} \quad D^T\vec{e} = 0.$$

Now, since $\vec{e} = \vec{y} - D\vec{p}$ from (6), $D^T\vec{e} = 0$ means

$$D^T(\vec{y} - D\vec{p}) = 0 \quad \Rightarrow \quad D^T D\vec{p} = D^T\vec{p}.$$

In particular, when $D^T D$ is invertible, the least squares estimate is:

$$\hat{\vec{p}} = (D^T D)^{-1} D^T \vec{y}.$$

Returning to the problem of system identification, let's first consider the scalar system:

$$x(t+1) = \lambda x(t) + bu(t) + e(t)$$

where $e(t)$ is a disturbance term. It follows that

$$\begin{aligned} x(1) &= \lambda x(0) + bu(0) + e(0) \\ x(2) &= \lambda x(1) + bu(1) + e(1) \\ &\vdots \\ x(\ell) &= \lambda x(\ell-1) + bu(\ell-1) + e(\ell-1) \end{aligned}$$

where ℓ is the number of measurements. Rewriting the above as

$$\underbrace{\begin{bmatrix} x(0) & u(0) \\ x(1) & u(1) \\ \vdots & \dots \\ x(\ell-1) & u(\ell-1) \end{bmatrix}}_{=: D} \underbrace{\begin{bmatrix} \lambda \\ b \end{bmatrix}}_{=: \vec{p}} + \underbrace{\begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(\ell-1) \end{bmatrix}}_{=: \vec{e}} = \underbrace{\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(\ell) \end{bmatrix}}_{=: \vec{y}}$$

we obtain a standard Least Squares problem. Thus, when the 2×2 matrix $D^T D$ is invertible, we obtain the estimates $\hat{\lambda}$, $\hat{b}$ from

$$\hat{\vec{p}} = \begin{bmatrix} \hat{\lambda} \\ \hat{b} \end{bmatrix} = (D^T D)^{-1} D^T \vec{y}.$$

In practice $D^T D$ is invertible when the measurements contain enough information about the unknown parameters. A trivial scenario where $D^T D$ is not invertible is when we apply zero input $u(t) = 0$ for all t. In this case the measurements contain no information about the parameter b, and naturally the estimation problem is ill-posed.