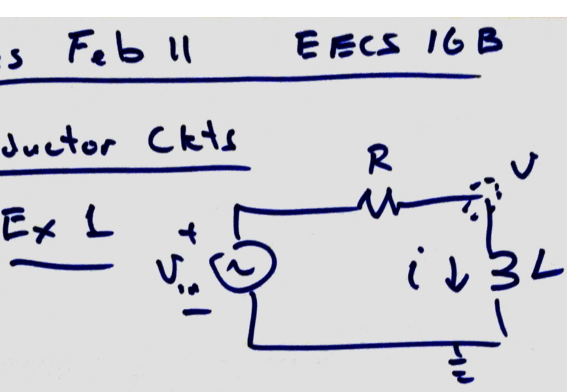
EECS 16B Feb 11 2020 Inductor Circuits and Complex Eigenvalues, Eigenvectors

Tuesday, February 11, 2020 8:11 PM

① Tues Feb 11 EECS 16B

* Inductor Ckts

Ex 1



$$\frac{d}{dt} i = -\frac{R}{L} i + \frac{1}{L} V_{in}$$

eigenvalue $\lambda = -\frac{R}{L}$

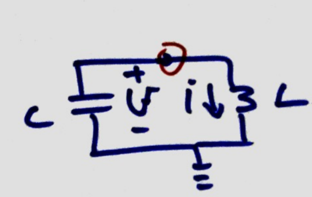
time constant $\tau = \frac{L}{R} \overset{\text{unit}}{=} \frac{V\text{-}s/A}{V/A} = s$

L = Henry's = $\frac{\text{Volt-sec}}{\text{amps}}$

$L = \frac{\lambda}{i}$ ← flux

Ex 2 L-C ckt - homogenous

②



Special Case
$L = 1H$
$C = 1F$

$$c \frac{d}{dt} v + i = 0$$

$$L \frac{d}{dt} i = v$$

$$\frac{d}{dt}\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} ; \quad \begin{bmatrix} v \\ i \end{bmatrix}(t_0) = ✓$$

$$\frac{d}{dt}\vec{x} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A}\vec{x} ; \quad \vec{x} = \begin{bmatrix} v \\ i \end{bmatrix}$$

③ Analysis - eigenvalues ; eigenvectors

eigen analysis : char. poly = $\det[\lambda I - A]$

$$\det\begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = 0 = \lambda^2 + 1$$

roots $\lambda_{1,2} = \pm j$; $j = \sqrt{-1}$

$\lambda_1 = j$; $\lambda_2 = -j$

eigenvectors

$\lambda_1 \leftrightarrow \begin{bmatrix} j & 1 \\ -1 & j \end{bmatrix}\vec{v}_1 = \vec{0}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix}$

$\lambda_2 \leftrightarrow \begin{bmatrix} -j & 1 \\ -1 & -j \end{bmatrix}\vec{v}_2 = \vec{0}$; $\vec{v}_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}$

④ $\lambda_2 = \overline{\lambda_1}$ and $\vec{v}_2 = \overline{\vec{v}_1}$

* eig values & eig vectors seem to come in conjugate pairs
* Why this structure?
  - true with real-valued A matrix

$\det(\lambda I - A)$

$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$

$a_0, a_1, \ldots, a_{n-1}$ are all real-valued since these are sums of products of entries of A matrix

⑤ Fund Thm of Algebra

* { n^{th} order polynomial has n roots over $\mathbb{C}$

{ With a real valued polynomial, roots are real, or occur in complex conj. pairs

product rule for conjugation

$$z_1 \cdot z_2 = |z_1| e^{j\theta_1} \cdot |z_2| e^{j\theta_2}$$

$$= |z_1||z_2| e^{j(\theta_1+\theta_2)}$$

$$\bar{z}_1 \cdot \bar{z}_2 = |z_1| e^{-j\theta_1} \cdot |z_2| e^{-j\theta_2}$$

⑥ $\bar{z}_1 \cdot \bar{z}_2 = |z_1||z_2| e^{-j(\theta_1+\theta_2)}$

$= \overline{(z_1 \cdot z_2)}$

return to our char. poly

$p(\lambda) = 0$ has all real coeff's, but λ_1 is complex

$$\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

$$\overline{p(\lambda)} = \overline{\lambda^n} + a_{n-1}\overline{\lambda^{n-1}} + \cdots + a_1\bar{\lambda} + a_0 = 0$$

$$= (\bar{\lambda})^n + a_{n-1}(\bar{\lambda})^{n-1} + \cdots + a_1\bar{\lambda} + a_0 = 0$$

⑦ Conclude : λ_1 is root $p(\lambda)$

say λ_1 is complex.

$\bar{\lambda}_1$ is also root of $p(\lambda)$

eigenvalues for real A : real or occur in conj. pairs

What about eigenvectors?

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\overline{(A\vec{v}_1)} = \overline{(\lambda_1 \vec{v}_1)} \Leftrightarrow A \cdot \overline{\vec{v}_1} = \bar{\lambda}_1 \cdot \overline{\vec{v}_1}$$

⇒ eigenvectors also in conj. pairs

⑧ Back to L-C example

$\lambda_1 = +j$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix}$; $\lambda_2 = -j$, $v_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}$

Homogeneous sol'n $\quad \vec{x}(0) = \begin{bmatrix} 1V \\ 0A \end{bmatrix}$

$x = \begin{bmatrix} v \\ i \end{bmatrix}$

$$\vec{x}(0) = \underbrace{\begin{bmatrix} 1 \\ -j \end{bmatrix}}_{\vec{v}_1}\frac{1}{2} + \underbrace{\begin{bmatrix} 1 \\ j \end{bmatrix}}_{\vec{v}_2}\frac{1}{2} \quad ✓$$

⑨ Solution

$$\vec{x}(t) = \frac{1}{2}\begin{bmatrix} 1 \\ -j \end{bmatrix} e^{jt} + \frac{1}{2}\begin{bmatrix} 1 \\ +j \end{bmatrix} e^{-jt}$$

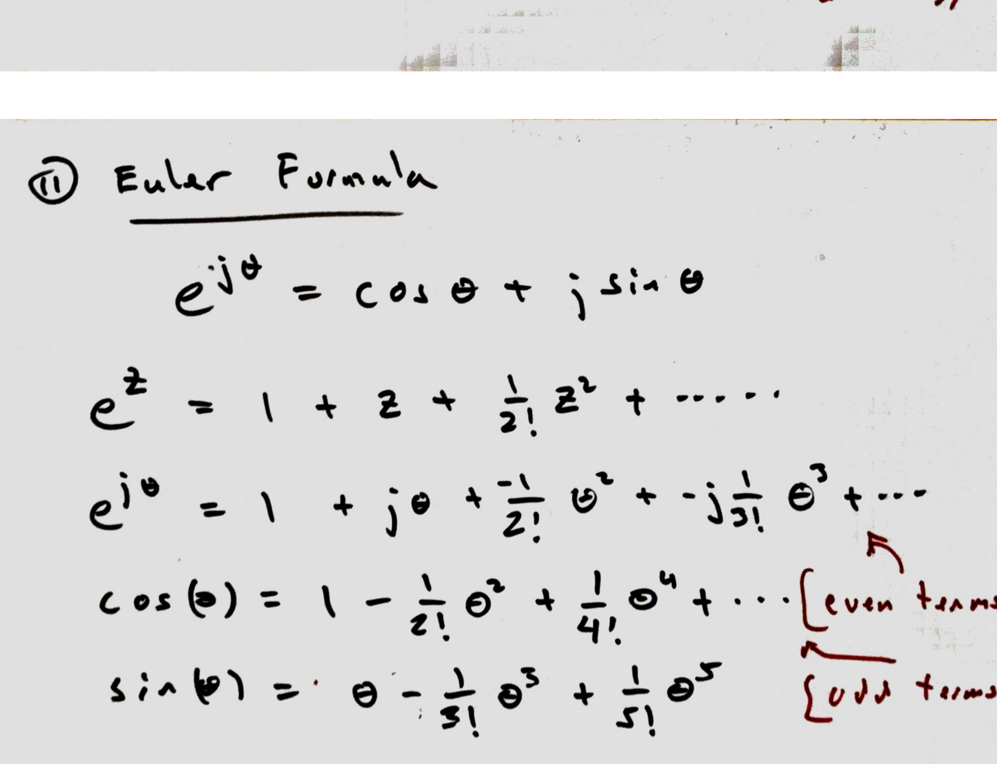
$$\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

$$\sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\vec{x}(t) = \begin{bmatrix} \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} \\ \frac{1}{2j}e^{jt} - \frac{1}{2j}e^{-jt} \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

⑩ Oscilloscope



$v^2 + i^2 = 1$

units: $cv^2 + Li^2 = \text{const.}$
$= 2 \cdot \text{Energy}$

⑪ Euler Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^z = 1 + z + \frac{1}{2!}z^2 + \cdots$$

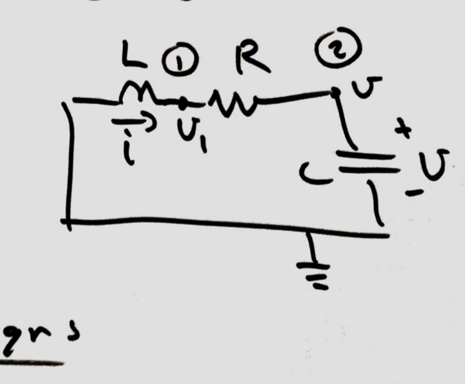
$$e^{j\theta} = 1 + j\theta + \frac{-1}{2!}\theta^2 + -j\frac{1}{3!}\theta^3 + \cdots$$

$$\cos(\theta) = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \cdots \text{ [even terms]}$$

$$\sin(\theta) = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 \text{ [odd terms]}$$

⑫ L-R-C ckt

Ex3:



Node eqns

① $-i + \frac{v_1 - v}{R} = 0$

② $\frac{v - v_1}{R} + C\frac{d}{dt}v = 0$

③ $L\frac{d}{dt}i = -v_1$

⑫ ① ⇒ ② replace v_1 in ②

② $C\frac{d}{dt}v = i$

③ $L\frac{d}{dt}i = -Ri - v$

$$\frac{d}{dt}\begin{bmatrix} v \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}_{A}\begin{bmatrix} v \\ i \end{bmatrix}$$

Analysis

$$0 = \det(\lambda I - A) = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

roots: $\lambda_{1,2} = -\frac{1}{2}\frac{R}{L} \pm \sqrt{\left(\frac{1}{2}\frac{R}{L}\right)^2 - \frac{1}{LC}}$

⑬ How things scale as R increases:

1) $R = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{-1/LC} = \pm j\sqrt{\frac{1}{LC}}$ ✓

↕

2) $\left(\frac{1}{2}\frac{R}{L}\right)^2 = \frac{1}{LC} \Rightarrow \lambda_{1,2} = -\frac{1}{2}\frac{R}{L}$

3) $\left(\frac{1}{2}\frac{R}{L}\right)^2 > \frac{1}{L} \Rightarrow \lambda_{1,2}$ are distinct & real-valued

Between (1) & (2):
complex conjugate eigenvalues:

$$\lambda_{1,2} = \underbrace{-\frac{1}{2}\frac{R}{L}}_{\lambda_r} \pm j\underbrace{\sqrt{\frac{1}{LC} - \left(\frac{1}{2}\frac{R}{L}\right)^2}}_{\lambda_i}$$

$\lambda_r \pm j\lambda_i$

⑭ Know we get $e^{\lambda t}$ as time response:

$$e^{(\lambda_r + j\lambda_i)t} = e^{\lambda_r t} e^{j\lambda_i t}$$

$$= e^{\lambda_r t}\left[\cos\lambda_i t + j\sin\lambda_i t\right]$$