EECS 16B Designing Information Devices and Systems II
Fall 2019

Note: Phasors

1 Overview

Frequency analysis focuses on analyzing the steady-state behavior of circuits with sinusoidal voltage and current sources — sometimes called AC circuit analysis. This note will show you how to do this more easily.

A natural question to ask is: what's so special about sinusoids? One aspect is that sinusoidal sources are very common - for instance, in the voltage output by a dynamo - making this form of analysis very useful. The most important reason, however, is that analyzing sinusoidal functions is *easy*! Whereas analyzing arbitrary input signals (like in transient analysis) requires us to solve a set of differential equations, it turns out that we can use a procedure very similar to the seven-step procedure from EE16A in order to solve AC circuits with only sinusoidal sources.

2 Scalar Linear First-Order Differential Equations

We've already seen that general linear circuits with sources, resistors, capacitors, and inductors can be thought of as a system of linear, first-order, differential equations with sinusoidal input. By developing techniques to determine the steady state of such systems in general, we can hope to apply them to the special case of circuit analysis.

First, let's look at the scalar case, for simplicity. Consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t),$$

where the input $u(t)$ is of the form

$$u(t) = ke^{st}$$

where $s \neq \lambda$.

We've previously seen how to solve this equation, and saw that

$$x(t) = \left(x_0 - \frac{k}{s-\lambda}\right)e^{\lambda t} + \frac{k}{s-\lambda}e^{st},$$

where $x(0) = x_0$ is a parameter depending on the initial value of $x(t)$.

The interesting thing about this solution is that it's almost a scalar multiple of $u(t)$ - if we ignore the initial term involving $e^{\lambda t}$, then $x(t)$ linearly depends on $u(t)$. It'd be nice if we could somehow ignore that initial term, by arguing that it goes to zero over time. Then, our "steady state" solution for $x(t)$ would involve only the e^{st} term, which seems to make our lives t lot easier.

When might that happen? Specifically, when does $e^{\lambda t} \to 0$ as $t \to \infty$? If λ were real, the answer is obvious - the term decays to zero if and only if $\lambda < 0$.

But what about for complex λ? We can try writing a complex λ in the form $\lambda = \lambda_r + j\lambda_i$, to try and reduce the problem to the real case. Then, we see that

$$\begin{aligned} e^{\lambda t} &= e^{(\lambda_r + j\lambda_i)t} \\ &= e^{\lambda_r t} e^{j\lambda_i t}. \end{aligned}$$

The $e^{\lambda_r t}$ is nice, since it's exactly the real case we just saw above. But what to do with the $e^{j\lambda_i t}$ term? Well, the only thing we can really do here is apply Euler's formula, so we find that

$$e^{\lambda t} = e^{\lambda_r t}(\cos(\lambda_i t) + j\sin(\lambda_i t)).$$

This expression seems promising! The first term in the product is a real exponential, which we know decays to zero exactly when $\text{Re}[\lambda] = \lambda_r < 0$. The second term is a sum of two sinusoids with unit amplitudes. Since the amplitude of each sinusoid is constant, their sum will clearly not decay to zero or grow to infinity over time. Thus, the asymptotic behavior of the overall expression is governed solely by the first term - $e^{\lambda t}$ will decay to zero exactly when e^{λ_r} does. Thus, applying our result from the real case, we see that $e^{\lambda t}$ goes to zero exactly when $\lambda_r < 0$.

Looking back at our solution for $x(t)$, we've now got a condition for when the $e^{\lambda t}$ decays that works for both real and complex λ.

3 Systems of Linear First-Order Differential Equations

Can we apply similar techniques to what we've just seen to a system of differential equations? Specifically, consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t),$$

where A is a fixed, real, matrix. As before, we will consider only control inputs of a special form, where each component is of the form ke^{st} for some constant k. More precisely, we will consider only inputs where $\vec{u}(t)$ can be expressed in the form

$$\vec{u}(t) = \vec{\tilde{u}}e^{st},$$

where $\vec{\tilde{u}}$ does not depend on t, and s is *not* an eigenvalue of the matrix A.

Inspired by our observations in the previous section, let's make the guess that our solution $x(t)$ can be written as

$$\vec{x}(t) = \vec{\tilde{x}}e^{st}$$

where $\vec{\tilde{x}}$ does not depend on t. Substituting into our differential equation, we find that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\tilde{x}}e^{st}) &= A\vec{\tilde{x}}e^{st} + \vec{\tilde{u}}e^{st} \\ \implies \quad s(\vec{\tilde{x}}e^{st}) &= A\vec{\tilde{x}}e^{st} + \vec{\tilde{u}}e^{st} \\ \implies \quad (sI - A)\vec{\tilde{x}}e^{st} &= \vec{\tilde{u}}e^{st}. \end{aligned}$$

Since the above equality must hold true for all t, it is clear that we can equate the coefficients of e^{st}, to obtain

$$(sI - A)\vec{\tilde{x}} = \vec{\tilde{u}}.$$

Now, recall that s was assumed not to be an eigenvalue of A. Imagine that $sI - A$ had a nonempty null space containing some vector $\vec{y}$. Then, by definition,

$$\begin{aligned} & (sI - A)\vec{y} = \vec{0} \\ \implies \quad & A\vec{y} = s\vec{y} \end{aligned}$$

so s would be an eigenvalue of A with corresponding eigenvector $\vec{y}$. Since we know that this cannot be the case, our initial assumption was flawed, so $sI - A$ must have an empty null space and so must be invertible.

Thus, we can rearrange our equation for $\vec{\tilde{x}}$ above, to express

$$\vec{\tilde{x}} = (sI - A)^{-1}\vec{\tilde{u}}.$$

It is straightforward to substitute this back into the expression for $x(t)$ and verify that it does indeed correspond to a valid solution for our original system of differential equations.

This is great! Starting with a system of differential equations with input of a particular form, we can now use the above identity to construct a solution for $\vec{x}(t)$ without calculus!

But is this solution the one we will reach in the steady state? Assume, for simplicity, that A has a full set of eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then we know that we can diagonalize A to be

$$\begin{aligned} A &= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} \\ &= V\Lambda V^{-1}, \end{aligned}$$

where V and Λ are the eigenvector and eigenvalue matrices in the above diagonalization.

Thus, we can rewrite our differential equation for $x(t)$ as

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t) + \vec{u}(t) \\ \implies \quad & \frac{\mathrm{d}}{\mathrm{d}t}(V^{-1}\vec{x}(t)) = \Lambda(V^{-1}\vec{x}(t)) + V^{-1}\vec{u}(t). \end{aligned}$$

As we have seen many times, this diagonalized system of differential equations can be rewritten as a set of scalar differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(V^{-1}\vec{x}(t))[i] = \lambda_i(V^{-1}\vec{x}(t))[i] + (V^{-1}\vec{u}(t))[i],$$

where the $[i]$ represents the ith component of the associated vector, and λ_i is the ith eigenvalue of A.

Since $(V^{-1}\vec{u}(t))[i]$ is a multiple of e^{st} and $s \neq \lambda_i$, we know from our scalar results that the solution to $(V^{-1}\vec{x}(t)))[i]$ can be expressed as a linear combination of $e^{\lambda t}$ and e^{st}, where the $e^{\lambda t}$ decays to zero over time if and only if $\mathrm{Re}[\lambda_i] < 0$, yielding a steady state solution involving only a scalar multiple of e^{st}. Let this solution be

$$(V^{-1}\vec{x}(t))[i] = \vec{\tilde{x}}[i]e^{st}.$$

Thus, we can stack these solutions for i and pre-multiply by V to obtain

$$\begin{aligned} V^{-1}\vec{x}(t) &= \vec{\tilde{x}}e^{st} \\ \implies \qquad \vec{x}(t) &= (V\vec{\tilde{x}})e^{st}. \end{aligned}$$

Now, recall that our candidate solution $\vec{x}(t) = \vec{\tilde{x}}e^{st}$ was constructed to be the unique solution to our system that was a scalar multiple of e^{st}. Thus, our candidate solution is exactly the steady state solution to the system, which we will converge to exactly when the real components of all the eigenvalues λ_i of our state matrix A are less than zero.

4 Circuits with Exponential Inputs

Now we know how to handle systems of differential equations with exponential inputs, let's see how to use these techniques to analyze physical circuits.

Imagine that we have a large circuit involving resistors, capacitors, and inductors, driven by inputs linearly dependent on the exponential function e^{st} for some constant s. From our study of transient analysis, we know that this circuit's behavior can be described by the linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t),$$

where $\vec{u}(t) = \tilde{u}e^{st}$ for some $\tilde{u}$ independent of time, and specific voltages (across capacitors) and currents (through inductors) form the natural components of $\vec{x}(t)$.

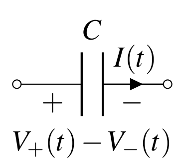
From our above results, we know that the steady state of this system will be of the form

$$x(t) = \tilde{x}e^{st},$$

so all the node voltages and elementwise currents of the circuit will be linearly dependent on e^{st} in the steady state.

The question is whether we actually have to simplify everything down to a differential equation at all? After all, the system of differential equation only arose because the capacitor and inductor element equations gave rise to derivatives. But what if we use our e^{st} insights at the level of the circuit element equations themselves?

Now, consider a particular capacitor C within the circuit, with node voltages $V_+(t)$ and $V_-(t)$ at its two terminals and a current $I(t)$ flowing through it.:



At steady state, we know from our understanding of the differential equation story above that

$$V_+(t) = \widetilde{V_+}e^{st}, V_-(t) = \widetilde{V_-}e^{st}, \text{ and } I(t) = \tilde{I}e^{st}$$

for some constants $\widetilde{V_+}$, $\widetilde{V_-}$ and $\tilde{I}$.

By the known differential equation for a capacitor, we have that

$$\begin{aligned} I(t) &= C\frac{\mathrm{d}}{\mathrm{d}t}(V_+(t) - V_-(t)) \\ \implies \quad \widetilde{I}e^{st} &= C\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{V_+}e^{st} - \widetilde{V_-}e^{st}) \\ \implies \quad \widetilde{I}e^{st} &= Cs(\widetilde{V_+} - \widetilde{V_-})e^{st} \\ \implies \quad \widetilde{I} &= Cs(\widetilde{V_+} - \widetilde{V_-}). \end{aligned}$$

Critically, this equation has no time-dependence - it is a purely linear equation relating some components of $\widetilde{x}$!

Similar equations can be obtained (this is a useful exercise to do) for an inductor $\widetilde{V}_+ - \widetilde{V}_- = \widetilde{I}Ls$. and a resistor $\widetilde{V}_+ - \widetilde{V}_- = \widetilde{I}R$. Rewriting the capacitor relationship to be in the same form, we see $\widetilde{V}_+ - \widetilde{V}_- = \widetilde{I}(\frac{1}{Cs})$. This suggests that we can view capacitors, inductors, and resistors as all being similar. In effect, capacitors and inductors just have s-dependent resistances. These are called s-impedances. A capacitor has an s-impedance of $\frac{1}{Cs}$. An inductor has an s-impedance of Ls. And a resistor's s-impedance is just the same as its resistance R.

This reveals an approach for circuit analysis with exponential inputs, as long as all the inputs have the same s. We can replace all the independent voltage and current sources with constant voltages and currents corresponding to the coefficients of e^{st}. We just replace all capacitors and inductors with their corresponding s-impedances, and then just analyze the entire circuit as though it only had resistances it it. This can be solved using Gaussian elimination or any other technique from 16A, and we can interpret the results to get the steady-state solution for the system.

5 Sinusoids

Unfortunately, there's one big issue with all the work we've done so far - specifically, the restrictions we imposed on our input $\vec{u}(t)$. We stated that $\vec{u}(t)$ should be expressed as

$$\vec{u}(t) = \vec{\widetilde{u}}e^{st}$$

for some s. What kinds of s are probably useful? If $\mathrm{Re}(s) < 0$, then we know that the input approaches zero over time, so the steady state behavior of our system is probably not very interesting. Similarly, if $\mathrm{Re}(s) > 0$, then our input will grow to infinity over time, so our state will blow up! This only leaves the case $\mathrm{Re}(s) = 0$ as neither blowing up or decaying away.

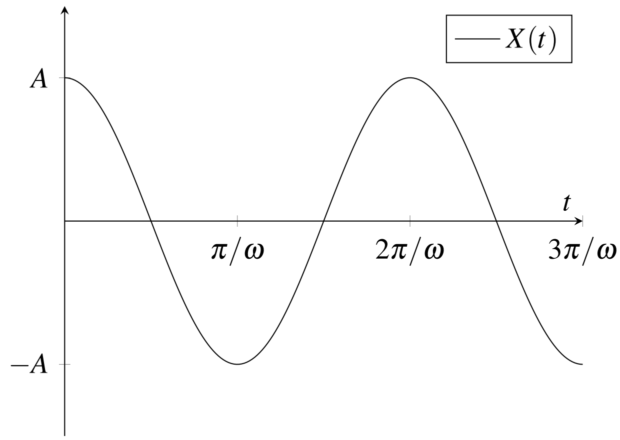
So then what can our input look like? If $\mathrm{Re}(s) = 0$, then s must be purely imaginary. So our input will be a linear function of e^{st}, where st is a real multiple of j. From Euler's formula, we know that term has some sort of periodic, sinusoidal behavior.

Now, in our circuit differential equations, the $u(t)$ is typically of the form

$$u(t) = A\cos(\omega t + \phi),$$

since it arises from the output of the AC current and voltage sources in our circuit. So a purely imaginary s seems ideal!

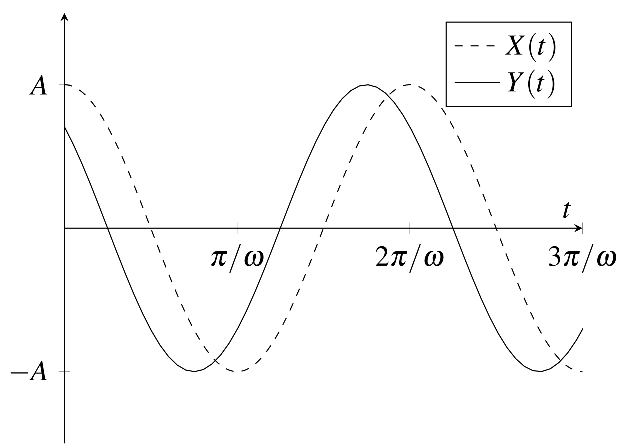
First, let's establish some terminology for sinusoids. Consider the function $X(t) = A\cos\omega t$:



You can think of X as representing an alternating current or voltage.

There are a couple properties of X that are immediately apparent from the figure: we call the maximum value of X above the mean (in this case, the x-axis) the *amplitude* (A), and the spacing between repetitions of the function the *period* ($T = 2\pi/\omega$).

However, there's one other important property of sinusoids: their *phase*. Consider the function $Y(t) = A\cos(\omega t + \phi)$.



Here, ϕ represents the *phase shift* of Y with respect to X. As can be seen, a positive phase shift moves the function to the left by that amount. In particular, notice that the sine and cosine functions are really the same sinusoid, with each just the other after a $\pi/2$ radian phase shift in the appropriate direction.

Now we know a little about sinusoids, let's see how we can rewrite them in terms of exponential functions. To do this, we will use complex numbers. We can combine Euler's formula with the properties of complex conjugates to determine that

$$e^{j\theta} + e^{-j\theta} = (\cos\theta + j\sin\theta) + (\cos\theta - j\sin\theta) = 2\cos\theta.$$

In other words, starting with two complex exponentials, we have pulled out a purely real sinusoid!

Let's see if we can use a similar sequence of algebraic manipulations to express an arbitrary sinusoidal input

$u(t) = A\cos(\omega t + \phi)$ in terms of exponential functions. From the above, we have that

$$\begin{aligned}
\cos(\theta) &= \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta} \\
\implies \quad \cos(\omega t + \phi) &= \frac{1}{2}e^{j\omega t + j\phi} + \frac{1}{2}e^{-j\omega t - j\phi} \\
&= \frac{e^{j\phi}}{2}e^{j\omega t} + \frac{e^{-j\phi}}{2}e^{-j\omega t} \\
\implies \quad u(t) &= A\cos(\omega t + \phi) \\
&= \frac{Ae^{j\phi}}{2}e^{j\omega t} + \frac{Ae^{-j\phi}}{2}e^{-j\omega t}.
\end{aligned}$$

Therefore, we can express an arbitrary sinusoid as a linear combination of two exponential functions! Notice that the coefficients of the two exponential functions are complex conjugates of one another. Thus, we can rewrite the above as

$$u(t) = \frac{Ae^{j\phi}}{2}e^{j\omega t} + \frac{\overline{Ae^{j\phi}}}{2}e^{-j\omega t}.$$

Thus, the coefficient of the $e^{j\omega t}$ can be used to represent the entire sinusoid $u(t)$ (assuming the frequency ω is known). We call this coefficient the *phasor* representing $u(t)$, and denote it as

$$\widetilde{u} = \frac{Ae^{j\phi}}{2}.$$

Now, we know how to find the steady states of systems of differential equations with sinusoidal inputs! First, use the above transformation to write the input as a linear combination of exponential functions e^{st}. Then, for each exponential function, solve the equation $\vec{\widetilde{x}} = (sI - A)\vec{\widetilde{u}}$ derived above to determine the steady state solution $\vec{x}(t) = \vec{\widetilde{x}}e^{st}$. Finally, take the superposition of all these steady states, to obtain the steady state corresponding to the entire original input!

This approach works great! But there's just one further optimization we can add, to simplify calculations further. Let's consider the case when we are working with real, sinusoidal inputs of a fixed frequency ω. Then we know that our input can be represented as a linear combination of inputs of the forms $e^{j\omega t}$ and $e^{-j\omega t}$.

But, from our above construction, we know that this can't be just *any* linear combination! Specifically, the coefficients of $e^{j\omega t}$ and $e^{-j\omega t}$ must be complex conjugates of one another! Thus, we can write our input as

$$\vec{u}(t) = \vec{\widetilde{u}}e^{j\omega t} + \overline{\vec{\widetilde{u}}}e^{-j\omega t}.$$

Now, let our system be

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t).$$

When $\vec{u} = \vec{\widetilde{u}}e^{j\omega t}$, we know that the steady state $\vec{\widetilde{x}}_1 e^{j\omega t}$ for $\vec{x}(t)$ is such that

$$(j\omega I - A)\vec{\widetilde{x}}_1 = \vec{\widetilde{u}}.$$

Similarly, when $\vec{u} = \overline{\vec{\widetilde{u}}}e^{-j\omega t}$, we know that the steady state $\vec{\widetilde{x}}_2 e^{-j\omega t}$ for $\vec{x}(t)$ is such that

$$(-j\omega I - A)\vec{\widetilde{x}}_2 = \overline{\vec{\widetilde{u}}}.$$

Then, we take the superposition of these two solutions to find the overall steady state solution to be

$$\vec{x}(t) = \vec{\tilde{x}}_1 e^{j\omega t} + \vec{\tilde{x}}_2 e^{-j\omega t}.$$

This solution is alright. But it requires us to solve two linear equations to find both $\vec{\tilde{x}}_1$ and $\vec{\tilde{x}}_2$, even though they appear to be fairly similar quantities. Can we somehow simplify our calculations further so that we can solve for $\vec{\tilde{x}}_2$ in terms of $\vec{\tilde{x}}_1$, so we only have to perform Gaussian elimination once?

The key observation to make is that

$$\overline{j\omega I - A} = -j\omega I - A,$$

since A is a real matrix. Thus, starting from the known solution for $\vec{\tilde{x}}_1$, we can take complex conjugates to obtain

$$\begin{aligned} (j\omega I - A)\vec{\tilde{x}}_1 &= \vec{\tilde{u}} \\ \implies \quad \overline{(j\omega I - A)}\overline{\vec{\tilde{x}}_1} &= \overline{\vec{\tilde{u}}} \\ \implies \quad (-j\omega I - A)\overline{\vec{\tilde{x}}_1} &= \overline{\vec{\tilde{u}}}. \end{aligned}$$

The above equation is exactly the equation that $\vec{\tilde{x}}_2$ has to satisfy. Thus, we see that

$$\vec{\tilde{x}}_2 = \overline{\vec{\tilde{x}}_1},$$

so we can substitute and write our final solution for $x(t)$ as

$$\vec{x}(t) = \vec{\tilde{x}}_1 e^{j\omega t} + \overline{\vec{\tilde{x}}_1} e^{-j\omega t}.$$

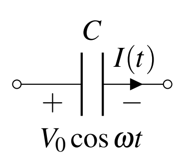
So only one round of Gaussian elimination (or linear equation system solving generally) is needed, not two!

6 Phasors

In principle, at this point we already know what to do when given a circuit with sinusoidal inputs all at the same frequency. But it can be helpful to make sure that you understand the derivations.

Let's apply the technique we've just developed to study the steady-state behavior of capacitors and inductors when supplied with a sinusoidal voltage or current signal. The key insight is that, under the condition that $\mathrm{Re}[\lambda_i] < 0$ for all eigenvalues λ_i of the state matrix corresponding to the circuit, we've shown that our steady state solution will only consist of terms with a linear dependence on terms of the form e^{st}, where our input linearly depends only on those same terms. In particular, when *all* the sources in our circuit operate at some angular frequency ω, all the other (real) components of our circuit's state will have to be real quantities represented as linear combinations of $e^{j\omega t}$ and $-e^{-j\omega t}$. But such linear combinations must also be sinusoids of frequency ω! Thus, every component of our state will be such a sinusoid!

Let's look at a capacitor provided with the sinusoidal voltage $V(t) = V_0 \cos(\omega t + \phi)$, as shown:



Note that we aren't assuming anything about the origin of the $V(t)$ - it could come from a voltage supply directly, or from some other complicated circuit. We see that we can write

$$V(t) = \frac{V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2} e^{-j\omega t},$$

so it can be represented by the phasor

$$\widetilde{V} = \frac{V_0 e^{j\phi}}{2}.$$

Now, by the capacitor equation, we know that

$$\begin{aligned} I(t) &= C \frac{\mathrm{d}}{\mathrm{d}t} V(t) \\ &= C \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2} e^{-j\omega t} \right) \\ &= C \left(\frac{j\omega V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{-j\omega V_0 e^{-j\phi}}{2} e^{-j\omega t} \right) \\ &= \frac{j\omega C V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{-j\omega C V_0 e^{-j\phi}}{2} e^{-j\omega t}, \end{aligned}$$

so we can represent the current as the phasor

$$\widetilde{I} = \frac{jC\omega V_0 e^{j\phi}}{2} = (j\omega C)\widetilde{V}.$$

In other words, having already shown that all steady state circuit quantities will be sinusoids with frequency ω, we now in fact can relate the phasors of the voltage across and the current through a capacitor by a ratio that depends only on the frequency and the capacitance.

This is exactly the same as the s-impedance story we told earlier. Because of this, when dealing with sinusoidal inputs at frequency ω, we use $s = +j\omega$ and just call the s-impedance, the *impedance*. The $+j\omega$ is understood from context.

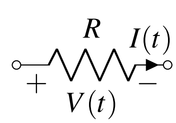
As before, this can be thought of as the "resistance" of a capacitor, since it relates the phasor representations voltage and current over and through the element by a constant ratio. For a capacitor, the impedance is

$$Z_C = \frac{\widetilde{V}}{\widetilde{I}} = \frac{1}{j\omega C}.$$

The interesting fact is that the impedance for the capacitor is imaginary.

We will now quickly perform a similar analysis for inductors and resistors.

Imagine some resistor R as follows:



Let $V(t)$ be represented by some phasor $\widetilde{V}$. Thus, by Ohm's Law,

$$\begin{aligned} & V(t) = \widetilde{V} e^{j\omega t} + \overline{\widetilde{V}} e^{-j\omega t} \\ \implies \quad & I(t) = \frac{1}{R} V(t) \\ & \quad = \frac{\widetilde{V}}{R} e^{j\omega t} + \frac{\overline{\widetilde{V}}}{R} e^{-j\omega t}, \end{aligned}$$

so we may represent the output current with the phasor

$$\widetilde{I} = \frac{\widetilde{V}}{R},$$

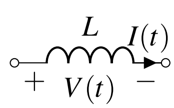
so the impedance is clearly

$$Z_R = R.$$

From this, we see that the impedance behaves very much like the resistance does, except that it generalizes to other circuit components.

Now, we will consider inductors. From our previous consideration of complex numbers, we have seen that any sinusoidal function can be represented by a phasor. Since we know that our steady states will all be sinusoids with the same frequency ω, we can start with a sinusoidal current and work in the opposite direction to calculate the impedance of an inductor, as follows.

Consider an inductor with voltage and current across it as follows:



Let the current $I(t)$ be represented by some phasor $\widetilde{I}$. Thus, by the equation of an inductor,

$$\begin{aligned} & I(t) = \widetilde{I} e^{j\omega t} + \overline{\widetilde{I}} e^{-j\omega t} \\ \implies \quad & V(t) = L \frac{\mathrm{d}I}{\mathrm{d}t} \\ & \quad = j\omega L \widetilde{I} e^{+j\omega t} - j\omega L \overline{\widetilde{I}} e^{-j\omega t} \\ & \quad = j\omega L \widetilde{I} e^{+j\omega t} + \overline{j\omega L \widetilde{I}} e^{-j\omega t}, \end{aligned}$$

so the voltage can be represented by the phasor

$$\widetilde{V} = j\omega L \widetilde{I}.$$

Thus, the impedance of an inductor is

$$Z_L = j\omega L.$$

7 Circuit Analysis

At this point, observe that we have essentially obtained "equivalents" to Ohm's Law for inductors and capacitors, using the impedance to relate their voltage and current phasors.

We will now try to show that a sum of sinusoidal functions is zero if and only if the sum of the phasors of each of those functions equals zero as well, to obtain a sort of "phasor-version" of KCL. Consider the sinusoids represented by the phasors

$$\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n.$$

Let $I_k(t)$ be the sinusoid represented by the phasor $\widetilde{I}_k$.

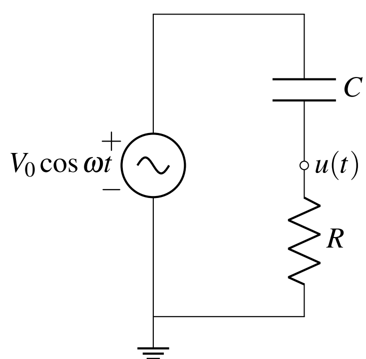
Observe that

$$\begin{aligned} & \widetilde{I}_1 + \widetilde{I}_2 + \ldots + \widetilde{I}_n = 0 \\ \iff \quad & (\widetilde{I}_1 + \widetilde{I}_2 + \ldots + \widetilde{I}_n)e^{j\omega t} = 0 \\ \iff \quad & (\widetilde{I}_1 + \widetilde{I}_2 + \ldots + \widetilde{I}_n)e^{j\omega t} + \overline{(\widetilde{I}_1 + \widetilde{I}_2 + \ldots + \widetilde{I}_n)}e^{-j\omega t} = 0 \\ \iff \quad & \sum_{k=1}^{n} \widetilde{I}_k e^{j\omega t} + \overline{\widetilde{I}_k} e^{-j\omega t} = 0 \\ \iff \quad & I_1(t) + I_2(t) + \ldots + I_n(t) = 0, \end{aligned}$$

so we have proved that a sum of sinusoids is zero if and only if the sum of their corresponding phasors is zero as well. This result can be thought of as a generalization of KCL to phasors.

Putting everything together, we have now successfully generalized all of our techniques of DC analysis to frequency analysis. We can finally consider some basic circuits, to verify that our technique works correctly.

Consider a voltage divider, where instead of one resistor we introduce a capacitor, as follows:



We are interested in knowing how the voltage $u(t)$ varies over time. Recall that we proved the voltage divider equation in the context of DC circuit analysis. However, that proof carries over to the phasor domain in a straightforward manner. Thus, the phasor $\widetilde{u}$ representing the voltage $u(t)$ can be represented in terms of the phasor $\widetilde{V}$ representing the supply voltage as follows:

$$\widetilde{u} = \frac{Z_R}{Z_C + Z_R}\widetilde{V},$$

where Z_C and Z_R are the impedances of the capacitor and resistor, respectively. Note also that, since the supply is at frequency ω, all other voltages and currents in the system will also be at the same frequency ω. Thus, using our results from earlier, we know that

$$Z_C = \frac{1}{j\omega C}$$

and

$$Z_R = R.$$

Note also that $\widetilde{V} = V_0/2$, using our equation for $\cos\theta$ from earlier.

Substituting these values into our equation for $\widetilde{u}$, we find that

$$\widetilde{u} = \frac{R/2}{\frac{1}{j\omega C} + R} V_0.$$

Finally, we can plug this result straight back into our equation relating phasors to real-valued sinusoids, to obtain

$$\begin{aligned} u(t) &= \widetilde{u}e^{j\omega t} + \overline{\widetilde{u}}e^{-j\omega t} \\ &= 2|\widetilde{u}|\cos(\omega t + \angle\widetilde{u}) \\ &= \frac{V_0}{\sqrt{1 + 1/(\omega RC)^2}} \cos\big(\omega t + \text{atan2}(1, \omega RC)\big). \end{aligned}$$

A Warning

Be aware that in this course phasors are defined slightly differently from how it is often done elsewhere. Essentially, there is a factor of 2 difference.

In this course, we define the phasor representation $\widetilde{X}$ of a sinusoid $x(t)$ to be such that

$$x(t) = \widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}.$$

However, elsewhere, the phasor representation may be defined such that

$$x(t) = \frac{1}{2}(\widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}).$$

Our definition is more natural and aligns to what you will see in later courses when you learn about Laplace and Fourier transforms. This is because our definition arises from the mathematics, and the same spirit of definition works even when working with inputs of the form e^{st} where s is not a purely imaginary number.

But then why would anyone ever use the alternative, more common definition? Its main advantage is that the magnitude of the phasor equals the amplitude of the signal. For instance, if we have the signal $A\cos(\omega t + \phi)$, then the alternative definition yields the phasor $Ae^{j\phi}$, with magnitude A. In contrast, our definition yields the phasor $(A/2)e^{j\phi}$. The former definition is convenient when conducting physical observations - when using an oscilloscope, one can easily see[1] the amplitude A of a signal, not the half-amplitude $A/2$.

Furthermore, it turns out that there are some slight calculation advantages (i.e. it makes some formulas simpler) to the more common definition when working with power systems and power electronics, which you may see if you take the relevant upper-division EE courses. However, for the purposes of the scope of this course, our definition is simpler and easier to understand, so we will stick with it throughout.

Of course, if the mathematics is done correctly, there is no real difference between the two definitions, in

[1] Actually in practice, if there is a DC component to the circuit — i.e. there are some inputs that are constants too — then the easiest thing to see is the peak-to-peak swing of the voltage which corresponds to twice the amplitude. So even the more common definition often forces the person using it to have to divide by two.

that both describe the same physical behaviors. It is just easier to do the mathematics correctly with the definiton we use here.

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