

# EECS 16B    Designing Information Devices and Systems II

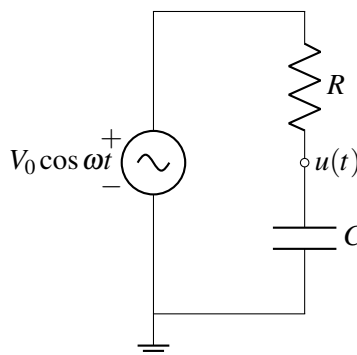
## Fall 2019    Note: Circuit Filters

### Overview

Using the method of phasors, we now know how to analyze the steady state of any AC circuit. We will now consider some applications of these circuits, and begin to explore techniques for designing these circuits to fit a set of requirements.

### 1 Circuit Example and Transfer Functions

Consider the circuit:



Notice that this circuit is very similar to a voltage divider, except that one resistor has been replaced by a capacitor. As we discussed last lecture, we can still apply our standard circuit analysis techniques on the phasors representing the voltages and currents in this system.

We saw last time that the input voltage phasor was

$$\tilde{V} = \frac{1}{2}V_0.$$

Applying the equation for a voltage divider, we find the voltage phasor at  $u(t)$  to be

$$\tilde{u} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tilde{V}.$$

We won't substitute in our known value for  $\tilde{V}$  just yet. We define the coefficient for  $\tilde{V}$  to be the *transfer function*

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}.$$

More broadly, transfer functions map an angular frequency  $\omega$  to a ratio of two phasors - an input and an output - and are used to characterize the behavior of a system across a range of frequencies. Here, our

transfer function relates two voltage phasors, but they can also relate two current phasors, or a voltage phasor with a current phasor, in a similar manner.

Observe that as transfer functions are the ratio of two phasors, they are complex-valued in general. Thus, we can write

$$H(\omega) = M(\omega)e^{j\theta},$$

where  $M(\omega)$  is the magnitude of the transfer function, and  $\theta$  is its argument.

In this example, we may rearrange  $H(\omega)$  to become

$$\begin{aligned} H(\omega) &= \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \\ &= \frac{1}{1 + j\omega RC} \\ &= \frac{1 - j\omega RC}{1 + \omega^2 R^2 C^2} \\ &= \frac{1}{1 + \omega^2 R^2 C^2} (1 - j\omega RC). \end{aligned}$$

Thus, since the coefficient at the front is real, we find that

$$\theta = \angle H(\omega) = \angle(1 - j\omega RC) = -\text{atan2}(\omega RC, 1).$$

We can also compute the magnitude of  $H$

$$M(\omega) = |H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}},$$

using the fact that  $|1/z| = 1/|z|$ .

Now, we can multiply through with  $\tilde{V}$ , to compute the voltage phasor  $\tilde{u}$

$$\begin{aligned} \tilde{u} &= H(\omega)\tilde{v} \\ &= M(\omega)e^{j\theta} \frac{V_0}{2} \\ &= \frac{V_0}{2\sqrt{1 + \omega^2 R^2 C^2}} e^{-j\text{atan2}(\omega RC, 1)}. \end{aligned}$$

Solving for  $u(t)$ ,

$$\begin{aligned} u(t) &= \tilde{u}e^{j\omega t} + \tilde{u}^*e^{-j\omega t} \\ &= \frac{1}{2} \left( V_0 M(\omega) e^{j\theta} e^{j\omega t} + V_0 M(\omega) e^{-j\theta} e^{-j\omega t} \right) \\ &= \frac{1}{2} \left( V_0 M(\omega) e^{j(\theta + \omega t)} + V_0 M(\omega) e^{-j(\theta + \omega t)} \right) \\ &= M(\omega) V_0 \cos(\omega t + \theta), \end{aligned}$$

which can be expressed using our solution for  $H$  as

$$u(t) = \frac{V_0}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t - \text{atan2}(\omega RC, 1)).$$

We see here that the transfer function will scale the input signal by a factor of  $M(\omega)$ , and shift its phase by an angle  $\theta$ . It can be easily shown that this property of transfer functions holds true for arbitrary sinusoidal input voltages, though we will not do so here.

Thus, we can follow the following procedure when analyzing a circuit accepting sinusoidal input signals:

- Express the sinusoidal input current or voltage as a phasor — the (generally complex) coefficient of  $e^{j\omega t}$  when we write the input currents or voltages as something times  $e^{j\omega t}$  plus something else times  $e^{-j\omega t}$ .
- Solve for the output voltage and current phasors as functions of the input phasor, and so determine the transfer function.
- Look at the magnitude and phase shift of the transfer function at the input frequency to determine the behavior of the circuit.

## 2 Filters

One very common use for AC circuits is as a *filter*. We have seen filters before briefly, with circuits blocking either low or high frequencies. The key idea behind filters is that of *superposition*. Recall, from EE16A, that in circuits consisting solely of sources and resistors, each independent source could be considered apart from the rest, with the final signal simply being the sum of all the intermediate responses from each of the sources.

Here, rather than looking at the superposition of independent sources, we will look at the superposition of input signals at the same point but with different frequencies. For notational convenience, let

$$G_{\omega}(\tilde{X}) = \tilde{X}e^{j\omega t} + \tilde{\tilde{X}}e^{-j\omega t}.$$

Basically,  $G_{\omega}(\tilde{X})$  converts the signal with angular frequency  $\omega$  represented by the phasor  $\tilde{X}$  from the frequency domain to the time domain.

Specifically, we assert that if we supply some input signal of the form

$$V_{in}(t) = \sum_i G_{\omega_i}(\tilde{X}_i)$$

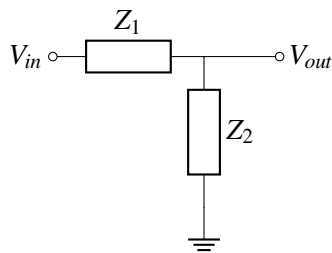
to a circuit with transfer function  $H(\omega)$ , the output voltage will be

$$V_{out}(t) = \sum_i G_{\omega_i}(H(\omega_i)\tilde{X}_i).$$

The above is an immediate consequence of the fact that all the differential equations that describe a single-input single-output AC system are linear.

The practical consequence of the above assertion is that we can treat the superposition of signals of two different frequencies as if the two signals were separate. For instance, if we supplied an AC circuit acting as a low-pass filter with the superposition of a 60 Hz signal and a 100 kHz noise signal, the high-frequency noise would be attenuated independently of the low-frequency signal.

Now, we will consider various filter configurations, and study their behavior. For now, we will look at *L filters* - filters consisting of two circuit elements placed in series, as shown:

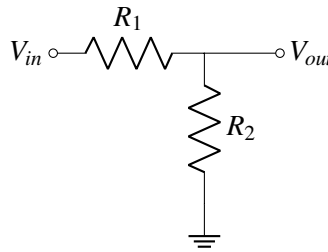


Recall from the impedance divider equation that the transfer function of this circuit is

$$H(\omega) = \frac{Z_2}{Z_1 + Z_2}.$$

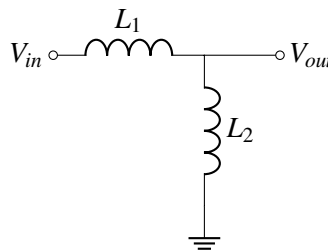
We will now consider all the possible forms of impedances that we can substitute into this equation. There are nine possible configurations, since we know of three different components in an AC circuit:

- Both components are resistors, as shown:



This forms a simple resistive voltage divider, which we have seen many times before, so we will not discuss it again here.

- Both components are inductors, as shown:



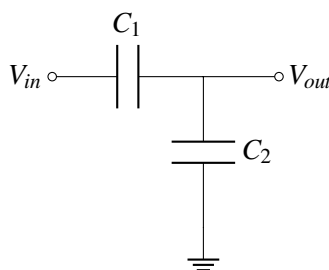
Substituting the known form for the impedance of an inductor into our equation for the transfer function, we obtain

$$H(\omega) = \frac{j\omega L_2}{j\omega L_1 + j\omega L_2} = \frac{L_2}{L_1 + L_2}.$$

Thus, this circuit exhibits behavior very similar to a resistive voltage divider, with its transfer function independent of frequency. Notice, however, that unlike resistors, inductors do not dissipate energy. A natural question to ask would be: what happens to the energy if we supply a DC voltage ( $\omega = 0$ ) to this circuit? Surely there will be some current flowing between the two potentials, which would cause energy to be dissipated, but where can this energy go? The resolution to this issue relies on the

observation that our expression for  $H(\omega)$  resolves to  $0/0$  for  $\omega = 0$ , since our simplifications assumed that  $\omega \neq 0$ . In the case of  $\omega = 0$ , it turns out that we can no longer model the inductors as being ideal - rather, their internal resistance will have an effect of dissipating energy.

- Both components are capacitors, as shown:

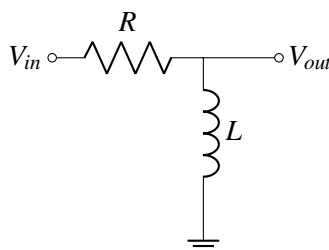


Substituting the known form for the impedance of a capacitor into our equation for the transfer function, we obtain

$$H(\omega) = \frac{\frac{1}{j\omega C_2}}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}} = \frac{C_1}{C_1 + C_2}.$$

Notice that here we again get something very similar to a voltage divider. Here, however, when  $\omega = 0$  no current is permitted to flow, since the capacitors act as open circuits. A charge sharing argument will nevertheless demonstrate that  $V_{out}$  will remain as we have claimed.

- One component is an inductor, and the other is a capacitor. We will discuss this circuit later, when we look at a phenomenon known as resonance.
- One component is a resistor, and the other is an inductor. There are two possible configurations of these two components. One of them is as follows:



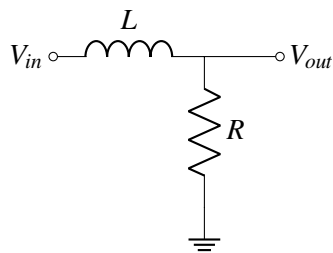
The above has the transfer function

$$H(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R}} = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c},$$

where  $\omega_c = 1/\tau = L/R$  is the reciprocal of the time constant  $\tau$  of an LR circuit, known as the *cutoff frequency*<sup>1</sup>. Why is this called the cutoff frequency? Because it turns out that this is around the frequency where the behavior of the filter seems to qualitatively change. Much above this frequency, the filter seems to let everything through. But much below this frequency, the filter seems to block everything. This kind of filter is called a “highpass” filter because it passes high frequencies.

The other configuration is as follows:

<sup>1</sup>Although we did not look at the transient analysis of LR circuits explicitly, the approach is very similar to that done in Lecture 2 for an RC circuit.

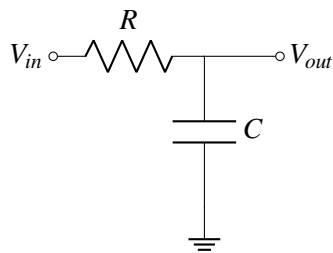


The above has the transfer function

$$H(\omega) = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}} = \frac{1}{1 + j\omega/\omega_c},$$

with  $\omega_c$  as defined previously. Notice how the qualitative behavior here is different. Here, frequencies that are much below  $\omega_c$  pass through unimpeded. While frequencies that are much above  $\omega_c$  largely get blocked. Such filters are called “lowpass” filters because they pass through low frequencies.

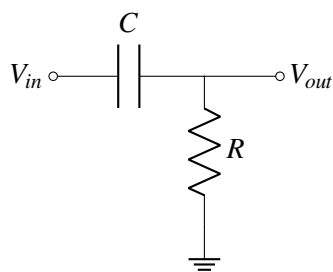
- One component is a resistor, and the other is an capacitor. There are two possible configurations of these two components. One of them is as follows:



The above has the transfer function

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega/\omega_c},$$

where  $\omega_c = 1/\tau$  as before. This time, however, since we have an RC-circuit,  $\tau = RC$ . The other configuration is as follows:



The above has the transfer function

$$H(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c},$$

with  $\omega_c$  as defined earlier.

That was a lot of filters! But looking closely at their transfer functions, excluding those filters behaving like a voltage divider, we see two distinct forms:

$$H(\omega) = \frac{1}{1 + j\omega/\omega_c}$$

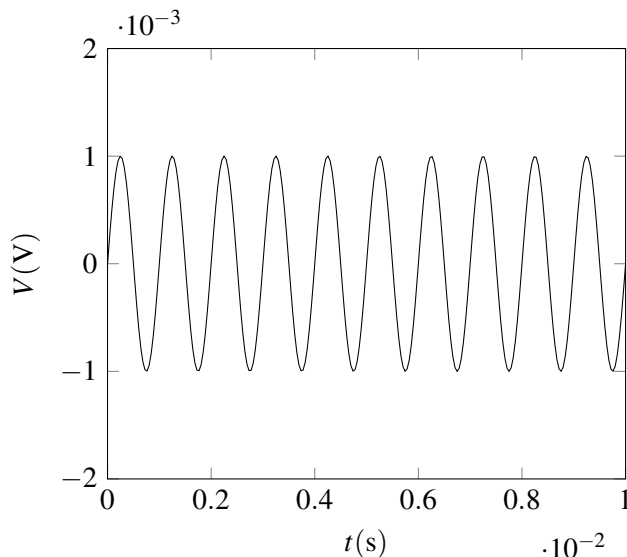
$$H(\omega) = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c}$$

Since the behavior of an AC circuit is solely determined by its transfer function, understanding the above two functions will be sufficient to analyze any of the above filters (except for the LC filters, which we will discuss in a future lecture).

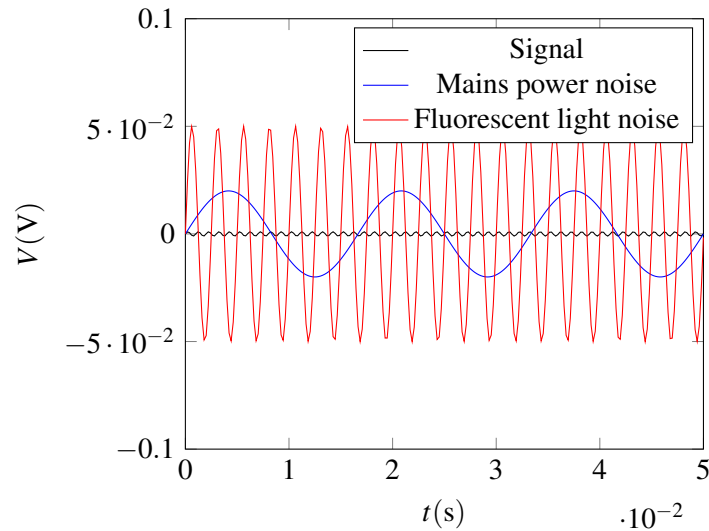
The first of these two transfer functions describes a *low-pass filter*, since for values of  $\omega \rightarrow 0$ ,  $H(\omega) \rightarrow 1$ , while for values of  $\omega \rightarrow \infty$ ,  $H(\omega) \rightarrow 0$ . In essence, low frequencies can pass through, while high frequencies are sent to ground. In a symmetric manner, the second of these transfer functions describes a *high-pass filter*, which has precisely the opposite behavior.

### 3 Design

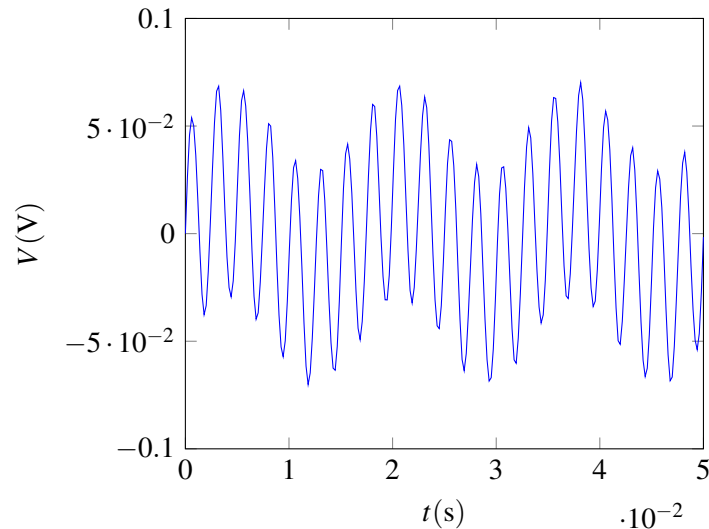
Now that we have a solid understanding of low- and high-pass filters, it's time to look at a toy example. Imagine that we have a signal with amplitude 1 mV and frequency approximately 1 kHz, as shown:



Now, we introduce some 60 Hz noise (perhaps from the mains power) at 20 mV, as well as some 100 kHz noise (perhaps from a malfunctioning fluorescent light ballast) noise at 50 mV, which are shown superimposed on our signal below:



Adding these noise sources to our signal, we obtain the resultant signal



At this point, our original signal is not even visible!

However, this is the 21st century! Someone might have told you about the magic of the Fourier transform<sup>2</sup>, which can break up a signal into its constituent sinusoids. All we need to do is record this signal precisely in a digital format, send it to a computer, and let some software library figure the rest out.

Unfortunately, we are encoding this signal using a 4-bit ADC, much like the one from lab. Recall that our ADC had a maximum voltage of 3.3 V, and had a least significant bit (in other words, a precision) of

$$\frac{3.3\text{ V}}{2^4} \approx 0.2\text{ V}.$$

But our signal only has an amplitude of 1 mV! What to do?

The natural response is to amplify our signal, perhaps using an op-amp circuit from EE16A, until the amplitude of our signal is of the same order as our measurement ability. To do so, we will need to amplify by

<sup>2</sup>We will actually deal with this at the end of 16B.



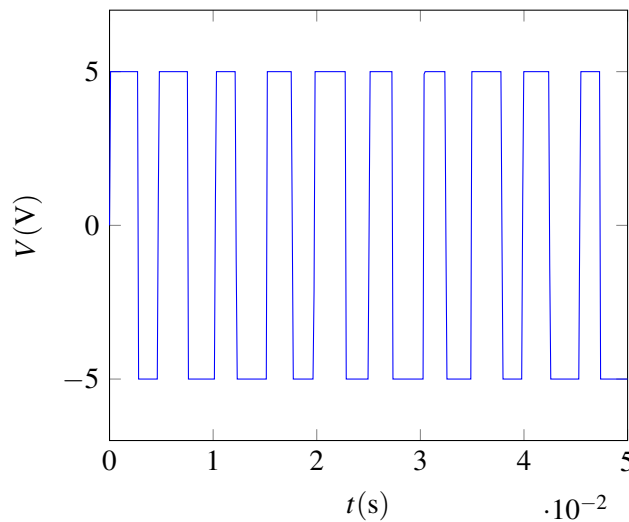
a factor of at least

$$\frac{3.3 \text{ V}}{1 \text{ mV}} = 3300.$$

Unfortunately, when we do this, our noise signals will shoot up, reaching amplitudes of

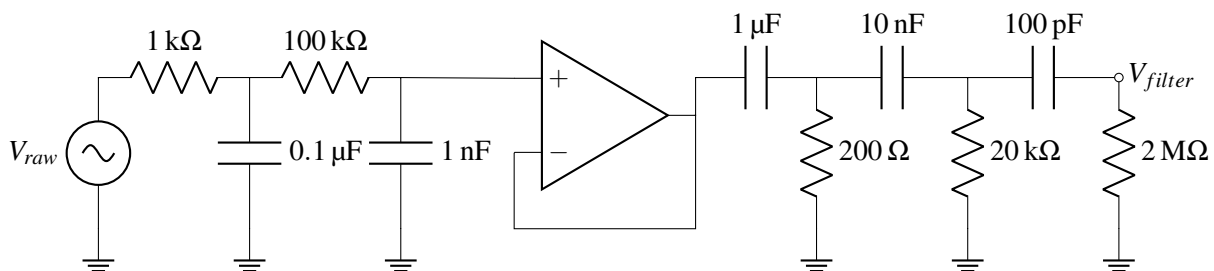
$$50 \text{ mV} \cdot 3300 = 165 \text{ V!}$$

Of course, our op-amps can only produce voltages between their rails, which we might set to  $\pm 5 \text{ V}$ . So our amplified signal will rail much of the time. Plotting the end result of this amplification, we obtain<sup>3</sup>:



Notice that our signal has railed for a significant portion of the cycle, meaning that information has been irretrievably lost before ever getting to a computer.

Clearly, analog processing is necessary in order to remove the noise from our circuit. What we'd like to do is remove both the high and low frequency noise from our circuit, while maintaining the amplitude of our signal frequency. We could then amplify the signal, without worrying about our op-amp railing. To do so, consider the following circuit:



The final amplification of  $V_{signal}$  is omitted for reasons of space, but can be easily constructed using an non-inverting amplifier.

Recall that our analysis of L-filters relied heavily on the voltage divider equation. That equation, in turn, assumed that all the current flowing into the top impedance flowed out of the bottom one. But here, some current flows out of the top impedance of each filter and into the next one in series. There are two ways to handle this issue. This first is to use a unity gain buffer, to allow us to access the output voltage of a

<sup>3</sup>The diagram is actually fairly inaccurate due to limitations of the plotting software - it should really look more like a solid block of color.

filter without drawing current. This is done here between the low-pass and high-pass filters, but is not done between all pairs of adjacent filters.

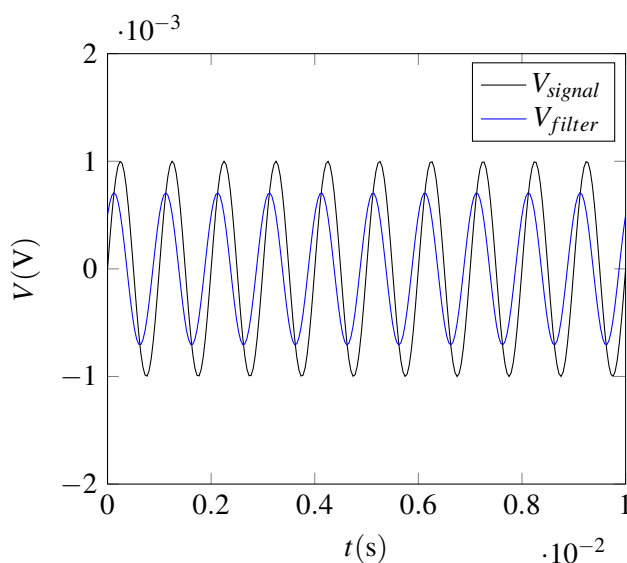
Instead, we simply try to make the current draw (known as the *load* of a filter) as low as possible, by increasing the impedance of subsequent filters. On the low-pass side, notice that the resistance of the resistor jumps by a factor of 100 from the first filter to the second, as does the impedance of the capacitors (as a capacitor's impedance is inversely proportional to its capacitance, reducing the capacitance actually increases the impedance). On the high-pass side, we again increase the impedance of each filter by a factor of 100 as well (for both the resistor and capacitor). Thus, each consecutive filter imposes only a very small load on the previous filters, so we can approximate the filters as being independent.

Of course, it may not be clear why chaining multiple low-pass or high-pass filters is necessary. Why not just use a single filter of each type? Notice that the transfer function  $H(\omega)$  of two filters chained together is the product of their two transfer functions  $H_1(\omega)$  and  $H_2(\omega)$ , assuming that the second filter draws very little current from the first. Consider the case of two low pass filters with time constant  $\tau = RC$  driven by a frequency significantly greater than  $\omega_p$ . Making linear approximations, observe that

$$\begin{aligned} |H_1(\omega)| &\approx \frac{1}{\omega RC} \\ |H_2(\omega)| &\approx \frac{1}{\omega RC} \\ |H(\omega)| &\approx \frac{1}{\omega^2 R^2 C^2}. \end{aligned}$$

Notice that  $|H(\omega)|$  drops with  $\omega^2$ , rather than  $\omega$ . If we were to plot it on a log-log plot, its slope would be  $-2$ , whereas  $H_1(\omega)$  and  $H_2(\omega)$  would only have a slope of  $-1$ . The magnitude of this slope is known as the *roll-off* of a filter. As the filter formed by chaining two low-pass filters together has a greater roll-off than either of the two filters did individually, it attenuates high-frequency signals much more, without significantly affecting signals of frequencies below  $\omega_p$  (since for those frequencies to first-order the transfer function remains  $\approx 1$ ). Similar behavior is present when chaining multiple high-pass filters in series, so it is advantageous to chain filters in this manner to reject undesirable frequencies more strongly.

Looking at the ultimate end-result  $V_{filter}$  versus  $V_{signal}$ , we observe

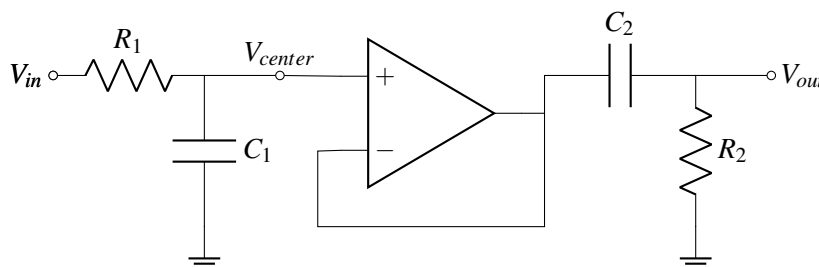


Notice that we have successfully used our low- and high-pass filters to reject undesired frequencies, but have attenuated our signal somewhat. Its phase has also shifted. Thus, we would have to adjust our amplification factor to compensate, but having done so we would be able to successfully measure our signal with just a 4-bit DAC.

## 4 Bode Plots

We will now consider a slightly more complex system and, in doing so, see the value in appropriate visualizations for transfer function behavior.

Imagine that two filters were cascaded in series, as follows:



Notice the op-amp serving as a unity gain buffer between the two filters. It is introduced to prevent the second circuit from loading the first, since we rely on no current being drawn from the intermediate node of the first filter, as we used the voltage divider equation in calculating its behavior.

The first filter is qualitatively a lowpass filter and the second one is a highpass one. What happens when we combine them?

Let the input voltage phasor be  $\tilde{V}_{in}$  at some frequency  $\omega$ , and let the two filters have cutoff frequencies  $\omega_{c_1}$  and  $\omega_{c_2}$  respectively. Thus, the voltage phasor at  $V_{center}$  is

$$\tilde{V}_{center} = H_1(\omega)\tilde{V}_{in}.$$

Since  $V_{center}$  is provided as an input to the second filter, we see that the output voltage phasor  $\tilde{V}_{out}$  is

$$\tilde{V}_{out} = H_2(\omega)\tilde{V}_{center} = H_2(\omega)H_1(\omega)\tilde{V}_{in}.$$

Thus, the net transfer function  $H(\omega)$  is

$$H(\omega) = H_1(\omega)H_2(\omega).$$

More generally, placing filters in series cascaded through buffers produces a circuit whose transfer function is the product of all the transfer functions of the constituent filters.

Recall that the product of two complex numbers has the properties that

$$\begin{aligned} |z_1z_2| &= |z_1||z_2| \\ \angle z_1z_2 &= \angle z_1 + \angle z_2. \end{aligned}$$

Thus, we see that

$$\log |z_1z_2| = \log |z_1| + \log |z_2|.$$

As a consequence, when plotting  $|H(\omega)|$  on a log-log plot, we can simply plot  $|H_1(\omega)|$  and  $|H_2(\omega)|$  and add them up. Similarly, when we plot  $\angle H(\omega)$ , we can again just add up  $\angle H_1(\omega)$  and  $\angle H_2(\omega)$ . We must be careful, however, to note that in most of our plots, the  $x$ -axis does *not* correspond to  $\theta$ , so we can't simply "stack" the two plots.

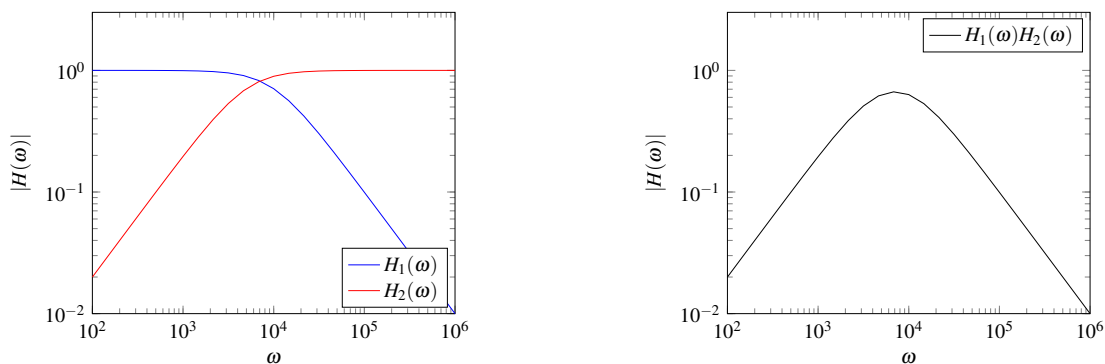
Such plots are called Bode plots. They are helpful in doing design.

Let's look at a numerical example, by setting the cutoff frequencies of the above filter as follows:

$$\omega_{c_1} = 10 \text{ kHz}$$

$$\omega_{c_2} = 5 \text{ kHz.}$$

Following this procedure for amplitude (with the individual filters on the left and the result on the right), we obtain



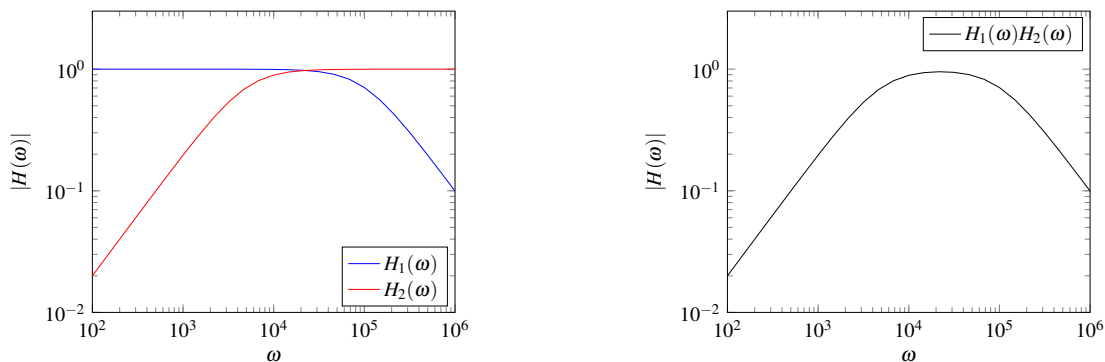
In the previous example, the cutoff frequencies were close to each other. Let's see what happens if we push them further apart.

Let's look at a numerical example, by setting the cutoff frequencies of the filter as follows:

$$\omega_{c_1} = 100 \text{ kHz}$$

$$\omega_{c_2} = 5 \text{ kHz.}$$

Following this procedure for amplitude (with the individual filters on the left and the result on the right), we obtain



Notice how the behavior is much more as we would like. We get both the highpass and lowpass effect, with a clean range of frequencies in the middle that seem to pass through. Such filters are called bandpass filters.

This construction can be used when the high frequencies we want to reject are well separated from the low

frequencies we want to reject.

If we wanted to make the filters sharper in how much they reject frequencies, we could cascade multiple copies of them together.

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