EECS 16B Designing Information Devices and Systems II
Fall 2019

Note: Controllability

1 Overview

We have seen how to model a continuous-time system[1] by choosing a time interval Δ, deciding to apply piecewise constant inputs (constant for durations of length $\Delta > 0$ seconds), and then sampling every Δ seconds to get a new[2] discrete time system:

$$\vec{x}(i+1) = A\vec{x}(i) + B\vec{u}(i),$$

where $\vec{x}(i)$ is the value of $\vec{x}_c(i\Delta)$ and $\vec{u}(i)$ is the value of $\vec{u}_c(t)$ for the time interval $t \in ((i-1)\Delta, i\Delta]$.

For the remainder of this module, we will almost entirely disregard the fact that our system may in fact really be continuous, focusing entirely on questions related to our discrete-time model. After all, in reality, most electronic control systems (like the MSP that we will use to develop a "robot car") have some intrinsic Δ in their ability to sample $\vec{x}$ and vary their output $\vec{u}$, so even if we knew that $\vec{x}$ varied within the interval Δ, we would not be able to measure or react to this variation within the time interval.

2 Special Cases of Controllability

A natural problem to address is that of *controllability* - in essence, given observations of $\vec{x}$ and some control over the input $\vec{u}$, how can we make $\vec{x}$ approach some target $\vec{x}^*$ over time? Clearly, if we have full control over $\vec{u}(i)$, this problem is easy! Given some $\vec{x}(0)$, we may choose $\vec{u}(0) = \vec{x}^* - A\vec{x}(0)$, to drive $\vec{x}(1)$ to our desired state in a single time step.

However, in reality, we seldom have that many degrees of freedom in our input $\vec{u}(i)$. Imagine, for instance, that our state equation modelled the behavior of a circuit connected to some input voltage that we can vary. We have seen previously that the state vector of a circuit may include many components, like the voltages across capacitors or the currents through inductors, none of which we can directly change. Therefore, we may use the following equation to obtain a more accurate model of our system:

$$\vec{x}(i+1) = A\vec{x}(i) + B\vec{u}(i).$$

This equation acknowledges that the number of inputs we can independently adjust at any single time might be different from the dimension of the state.

Notice the new matrix B, which constrains how our input $\vec{u}$ can influence the evolution of our system. For

[1] For example, defined by a system of differential equations $\frac{d}{dt}\vec{x}_c(t) = A_c\vec{x}_c(t) + B_c\vec{u}_c(t)$.

[2] The A matrix for the discrete time system is related to the original A_c matrix of the continuous-time system. It is easiest to see the exact relationship when there exists a coordinate system in which A_c is diagonal — $A_c = V\Lambda V^{-1}$. In the diagonal case $\frac{d}{dt}\vec{\widetilde{x}}_c(t) = \Lambda\vec{\widetilde{x}}_c(t) + \widetilde{B}_c\vec{u}_c(t)$, the underlying system is basically a parallel set of scalar differential equations $\frac{d}{dt}\widetilde{x}_c[j](t) = \lambda_j\widetilde{x}_c[j](t) + (\widetilde{B}_c\vec{u}_c)[j](t)$. If $\lambda_j \neq 0$, this discretizes to $\widetilde{x}[j](i+1) = e^{\Delta\lambda_j}\widetilde{x}[j](i) + \frac{e^{\Delta\lambda_j}-1}{\lambda_j}(\widetilde{B}_c\vec{u}(i))[j]$. If $\lambda_j = 0$, this discretizes to $\widetilde{x}[j](i+1) = \widetilde{x}[j](i) + \Delta\widetilde{B}_c\vec{u}(i)[j]$. The A and B matrices follow from this.

instance, in a simple system with a two-dimensional state, the case of

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

would mean that our input can only directly affect the second state. For now, we will not complicate matters further by considering nonlinear behavior or noise, so we can treat the above discrete-time equation as an accurate model of our system.

Before, we were always able to choose a $\vec{u}(0)$ to get to our target state in a single time step, no matter what the initial state $\vec{x}(0)$ was. Let's try doing the same thing now for arbitrary B:

$$\begin{aligned} & \vec{x}(1) = \vec{x}^* \\ \implies \quad & A\vec{x}(0) + B\vec{u}(0) = \vec{x}^* \\ \implies \quad & B\vec{u}(0) = \vec{x}^* - A\vec{x}(0). \end{aligned}$$

Observe here that we can only achieve our desired state if $\vec{x}^* - A\vec{x}(0) \in \text{range}(B)$, which is not always the case (since B may be a "tall" matrix). Let n be the dimension of $\vec{x}$ - in other words, let $\vec{x}$ be made up of n scalar components. For *any* desired state to be reachable, B would have to span all of $\mathbb{R}^n$, and so be of rank n.

So, is all hope lost? Not necessarily. Recall that we only wanted to achieve our target state at *some point*, not necessarily in a single time step. Consider the state transition matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix},$$

with B defined as before (i.e. only allowing us to affect the second state). Since B has only one column, our control input $\vec{u}$ is one dimensional, and so can be treated as a scalar u. We know that, after 1 time step, we may only reach the states that can be written as

$$\vec{x}(1) = A\vec{x}(0) + Bu(0)$$

for a suitable choice of control input $u(0)$. Since B is not of rank 2, the set of viable $\vec{x}(1)$ does not yet span all of $\mathbb{R}^2$. However, now consider the set of states that are reachable after 2 time steps. By our state transition equation, they can be written as

$$\begin{aligned} \vec{x}(2) &= A\vec{x}(1) + Bu(1) \\ &= A(A\vec{x}(0) + Bu(0)) + Bu(1) \\ &= A^2\vec{x}(0) + ABu(0) + Bu(1). \end{aligned}$$

Notice that if the vectors AB and B together span all of $\mathbb{R}^2$, then we would be able to choose coefficients $u(0)$ and $u(1)$ to reach *any desired* $\vec{x}(2)$, meaning that the system is controllable in 2 time steps.

3 General Controllability

Generalizing our equation for $\vec{x}(2)$ to the case of multidimensional control inputs (i.e. when B has k columns, not just 1) and arbitrarily large state dimensions n, we obtain

$$\vec{x}(2) = A^2\vec{x}(0) + AB\vec{u}(0) + B\vec{u}(1)$$

from an analogous calculation.

This can be reexpressed as

$$\vec{x}(2) = A^2\vec{x}(0) + \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} \vec{u}(0) \\ \vec{u}(1) \end{bmatrix},$$

using stacked matrix notation.

More generally, we know that

$$\vec{x}(i) = A^n\vec{x}(0) + \begin{bmatrix} B & AB & \cdots & A^{i-1}B \end{bmatrix} \begin{bmatrix} \vec{u}(i-1) \\ \vdots \\ \vec{u}(0) \end{bmatrix}.$$

Therefore, we find that our system is controllable after i time steps if and only if the matrix

$$\mathscr{C} = \begin{bmatrix} B & AB & \cdots & A^{i-1}B \end{bmatrix}$$

has a column span that is n-dimensional.

The question now becomes - is *every* system controllable after sufficiently many time steps? That is to say, is it true that, for any $n \times n$ matrix A and $n \times k$ matrix B, there exists some constant i such that the matrix $\begin{bmatrix} B & AB & \cdots & A^{i-1}B \end{bmatrix}$ is of full rank?

Clearly, this is not the case - after all, consider the case when the input has no effect on the state transition, when $B = 0$! But it may not be the case even in nontrivial cases - consider, for instance

$$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Observe that

$$AB = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$A^2B = \begin{bmatrix} 4 \\ 8 \end{bmatrix},$$

and so on, so all the columns of the stacked matrix will be linearly dependent. Thus, even though this system has what is inarguably a nontrivial state transition equation, it is still not controllable even after infinitely many time steps.

4 Determining Controllability

We will now attempt to devise a general method of determining whether or not a system is controllable. In essence, we want to know whether the infinite sequence

$$\text{range}(\begin{bmatrix} B \end{bmatrix}), \text{range}(\begin{bmatrix} B & AB \end{bmatrix}), \text{range}(\begin{bmatrix} B & AB & A^2B \end{bmatrix}), \ldots$$

will ever reach $\mathbb{R}^n$. If, at some finite point, this sequence reaches $\mathbb{R}^n$, then that's great, and we know that our system is controllable. However, if our system turns out to be uncontrollable, then this sequence will keep going on forever and will never reach $\mathbb{R}^n$. Clearly, it is not possible to somehow evaluate an infinite number of terms of our sequence. So then, with only a finite number of terms of the sequence, how can we ever confidently say that "no, this system is not controllable"?

Our approach will rely on the following key observation: that if all the columns of A^iB are linearly dependent on the previous A^jB for $j < i$, then all the columns of $A^{i+1}B$ will also be linearly dependent on the same set of A^jB. That is to say, if the ranges in the above sequence ever stop growing for even a single iteration, then they will never start growing again.

For simplicity, we will conduct our proof under the assumption that $B = \vec{b}$ is a column vector, so all the $A^iB = A^i\vec{b}$ have only one column. However, the exact same approach works in the most general case, except that the notation is a lot more messy.

By the condition of the observation and the definition of linear dependence, there exist coefficients α_j such that

$$A^i\vec{b} = \sum_{j=0}^{i-1} \alpha_j A^j\vec{b}.$$

Thus, we may write

$$\begin{aligned}
A^{i+1}\vec{b} &= A(A^i\vec{b}) \\
&= A\sum_{j=0}^{i-1} \alpha_j A^j\vec{b} \\
&= \sum_{j=0}^{i-1} \alpha_j A^{j+1}\vec{b} \\
&= \sum_{j=1}^{i} \alpha_{j-1} A^j\vec{b} \\
&= \alpha_{i-1}A^i\vec{b} + \sum_{j=1}^{i-1} \alpha_{j-1} A^j\vec{b} \\
&= \sum_{j=0}^{i-1} \alpha_{i-1}\alpha_j A^j\vec{b} + \sum_{j=1}^{i-1} \alpha_{j-1} A^j\vec{b} \\
&= \alpha_{i-1}\alpha_0\vec{b} + \sum_{j=1}^{i-1} (\alpha_{i-1}\alpha_j + \alpha_{j-1}) A^j\vec{b},
\end{aligned}$$

so $A^{i+1}B$ can be expressed as a linear combination of the same preceding A^jB.

With this result, we are on the road to developing an algorithm to determine the controllability of a system. Iteratively construct the sequence of ranges as described above. If the dimension of the ranges ever stops growing, we know that the dimension will never start to grow again as the sequence continues, so we know what possible $\vec{x}^*$ are reachable even after infinitely many time steps. The only issue is if the dimension of the ranges never stops growing. But this can never be the case, since the dimension of the ranges is bounded by n (since they are all subspaces of $\mathbb{R}^n$)!

Thus, after n iterations, the ranges will either have stopped growing during one of the intermediate iterations, or would have reached rank n and so will not be able to continue to grow thereafter.

The argument above was given for any single column $\vec{b}$ and so in spirit, it can apply to all the columns viewed one at a time. So, we might as well do them all together. Consequently, by considering the subspace

$$\text{range}(\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix})$$

we will be able to determine whether our system is controllable. If this range is of dimension n (and so spans the entirety of $\mathbb{R}^n$), then our system is controllable. Otherwise, it is not. Typically, we refer to this span as range($\mathscr{C}$), where $\mathscr{C}$ is defined as

$$\mathscr{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

An immediate consequence of this result is that a system is controllable if and only if it is controllable in at most n timesteps.

The next natural question is as follows: even if our system is controllable, how do we calculate the necessary inputs needed to reach a desired state $\vec{x}^*$? As it turns out, this is straightforward, if we consider the stacked matrix representation of $\vec{x}(n) = \vec{x}^*$:

$$\vec{x}^* = A^n\vec{x}(0) + \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \vec{u}(n-1) \\ \vdots \\ \vec{u}(0) \end{bmatrix} = A^n\vec{x}(0) + \mathscr{C} \begin{bmatrix} \vec{u}(n-1) \\ \vdots \\ \vec{u}(0) \end{bmatrix}.$$

Rearranging, we obtain

$$\mathscr{C} \begin{bmatrix} \vec{u}(n-1) \\ \vdots \\ \vec{u}(0) \end{bmatrix} = \vec{x}^* - A^n\vec{x}(0).$$

Using Gaussian elimination, we can now determine a sequence of control inputs to use, if such a sequence exists. Thus, we have resolved the problem that was initially posed at the beginning of our discussion. Later, we will learn how to choose potentially better sequences of control inputs if the solution above is not unique.

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