EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet

Discussion 4B

In this discussion, we're going to cover linear algebra concepts (Change of Basis, Diagonalization) that unlock powerful circuit analysis techniques going forward. We also introduce a new kind of circuit element called the "inductor"; Note 3B will be useful.

1. Coordinate Change of Basis: Examples

Consider a vector in the standard basis,

$$\vec{x} = a\vec{e}_1 + b\vec{e}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{I}\vec{x} \tag{1}$$

where, a, b are $\vec{x}$'s coordinates in the standard basis and $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the elementary standard basis vectors.

Given a new set of basis vectors, $\mathscr{V} = \{\vec{v}_1, \vec{v}_2\}$, if $\vec{x} \in \text{span}\{\mathscr{V}\}$, then we can find new coordinates in terms of this new basis. The new coordinates are called a_v, b_v and are described,

$$\vec{x} = a_v\vec{v}_1 + b_v\vec{v}_2 = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} = \mathbf{V}\vec{x}_v \tag{2}$$

Now consider another set of basis vectors, $\mathscr{U} = \{\vec{u}_1, \vec{u}_2\}$. If $\vec{x} \in \text{span}\{\mathscr{U}\}$, then we can find the coordinates of $\vec{x}$ in terms of this basis. These coordinates are called a_u, b_u and are described,

$$\vec{x} = a_u\vec{u}_1 + b_u\vec{u}_2 = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \mathbf{U}\vec{x}_u \tag{3}$$

All of these bases are equivalent representations of any vector $\vec{x} \in \mathbb{R}^2$; each with their own set of coordinates. The same logic can, of course, be extended to any number of dimensions.

$$\vec{x} = \begin{bmatrix} | & | \\ \vec{e}_1 & \vec{e}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} \tag{4}$$

$$\vec{x} = I\vec{x} = \mathbf{V}\vec{x}_v = \mathbf{U}\vec{x}_u \tag{5}$$

Now that we've seen a conceptual overview of the change-of-basis, we can proceed with the worksheet problems.

(a) *Transformation From Standard Basis To Another Basis in* $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$ where $\vec{x}_u$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{x}_v$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.

Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{x}_v$?

(b) *Transformation Between Two Bases in* $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases:

$$\mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_w = \mathbf{T}\vec{x}_v$. Let $\vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_w$. Repeat this for $\vec{x}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{x}_v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_w$?

2. Diagonalization

(a) Consider a matrix $\mathbf{A}$, a matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{A}$, and a diagonal matrix $\mathbf{\Lambda}$ with the eigenvalues of $\mathbf{A}$ on the diagonal (in the same order as the eigenvectors (or columns) of $\mathbf{V}$). From these definitions, show that

$$\mathbf{AV} = \mathbf{V\Lambda}$$

3. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage changes as a function of the derivative of the current across it. That is:

$$V_L(t) = L\frac{dI_L(t)}{dt}$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the equivalent "fundamental" circuit for an inductor:

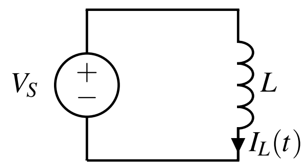

Figure 1: Inductor in series with a voltage source.

(a) What is the current through an inductor as a function of time? If the inductance is $L = 3$H, what is the current at $t = 6$s? Assume that the voltage source turns from 0V to 5V at time $t = 0$s, and there's no current flowing in the circuit before the voltage source turns on.

(b) Now, we add some resistance in series with the inductor, as in Figure 2.

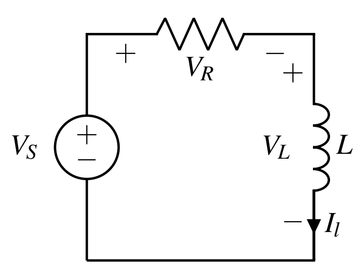

Figure 2: Inductor in series with a voltage source.

Solve for the current $I_L(t)$ in the circuit over time, in terms of R, L, V_S, t.

(c) **(Practice)** Suppose $R = 500\Omega, L = 1\text{mH}, V_S = 5\text{V}$. Plot the current through and voltage across the inductor ($I_L(t), V_L(t)$), as these quantities evolve over time.

4. Fibonacci Sequence

(a) The Fibonacci sequence is built as follows: the n-th number (F_n) is sum of the previous two numbers in the sequence. That is:

$$F_n = F_{n-1} + F_{n-2}$$

If the sequence is initialized with $F_1 = 0$ and $F_2 = 1$, then the first 11 numbers in the Fibonacci sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

We can express this computation as a matrix multiplication:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

What is $\mathbf{A}$?

(b) Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$.

(c) Diagonalize $\mathbf{A}$ (that is, in the expression $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, solve for each component matrix.)

(d) Use the diagonalized result to show that we can arrive at an analytical result for any F_n:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$

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