EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet Discussion 9B

1. Gram-Schmidt Algorithm

Let's apply Gram-Schmidt orthonormalization to a set of three linearly independent vectors $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$.

(a) Find unit vector $\vec{q}_1$ such that $\text{span}(\{\vec{q}_1\}) = \text{span}(\{\vec{s}_1\})$.

(b) Given $\vec{q}_1$ from the previous step, find $\vec{q}_2$ such that $\text{span}(\{\vec{q}_1, \vec{q}_2\}) = \text{span}(\{\vec{s}_1, \vec{s}_2\})$ and $\vec{q}_2$ is orthogonal to $\vec{q}_1$.

What would happen if $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ were *not* linearly independent, but rather $\vec{s}_1$ were a multiple of $\vec{s}_2$?

(c) Now given $\vec{q}_1$ and $\vec{q}_2$ in the previous steps, find $\vec{q}_3$ such that $\text{span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$, and $\vec{q}_3$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, and finally $\|\vec{q}_3\| = 1$.

(d) Let's extend this algorithm to n linearly independent vectors. That is, given an input $\{\vec{s}_1, \ldots, \vec{s}_n\}$, write the algorithm to calculate the orthonormal set of vectors $\{\vec{q}_1, \ldots, \vec{q}_n\}$, where $\text{span}(\{\vec{s}_1, \ldots, \vec{s}_n\}) = \text{span}(\{\vec{q}_1, \ldots, \vec{q}_n\})$.

Hint: How would you calculate the i^{th} vector, $\vec{q}_i$?

2. The Order of Gram-Schmidt

If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the set of vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Perform Gram-Schmidt on these vectors first in the order $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

(b) Now perform Gram-Schmidt on these vectors in the order $\vec{v}_3,\ \vec{v}_2,\ \vec{v}_1$. Do you get the same result?

Contributors:

- Regina Eckert.
- Druv Pai.