EECS 16B Spring 2021 — Designing Information Devices and Systems II — Discussion Worksheet

# Discussion 13A

This discussion will recap a lot of the key concepts covered in lecture last week.

## 1. Linear Approximation

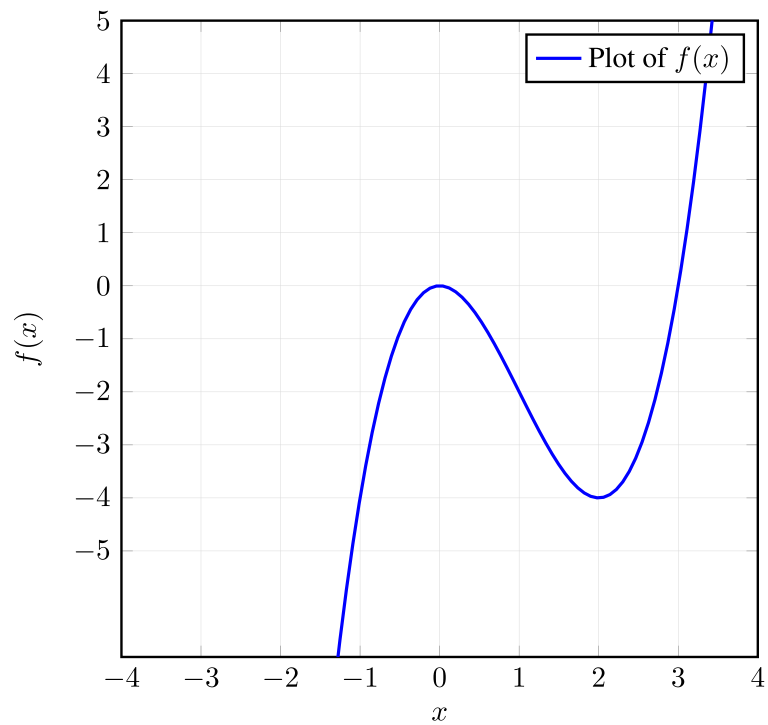
A common way to approximate a nonlinear function is to perform linearization near a point. In the case of a one-dimensional function $f(x)$, the linear approximation of $f(x)$ at a point $x_\star$ is given by

$$f(x) \approx f(x_\star) + f'(x_\star) \cdot (x - x_\star), \tag{1}$$

where $f'(x_\star) := \left.\frac{df(x)}{dx}\right|_{x=x_\star}$ is the derivative of $f(x)$ at $x = x_\star$.

Keep in mind that wherever we see $x_\star$, this denotes a *constant value* or operating point.

(a) Suppose we have the single-variable function $f(x) = x^3 - 3x^2$. We can plot the function $f(x)$ as follows:




i. Write the linear approximation of the function around an arbitrary point $x_\star$.

ii. Use the expression above to linearize the function around the point $x = 1.5$. Draw the linearization into the plot of part i).

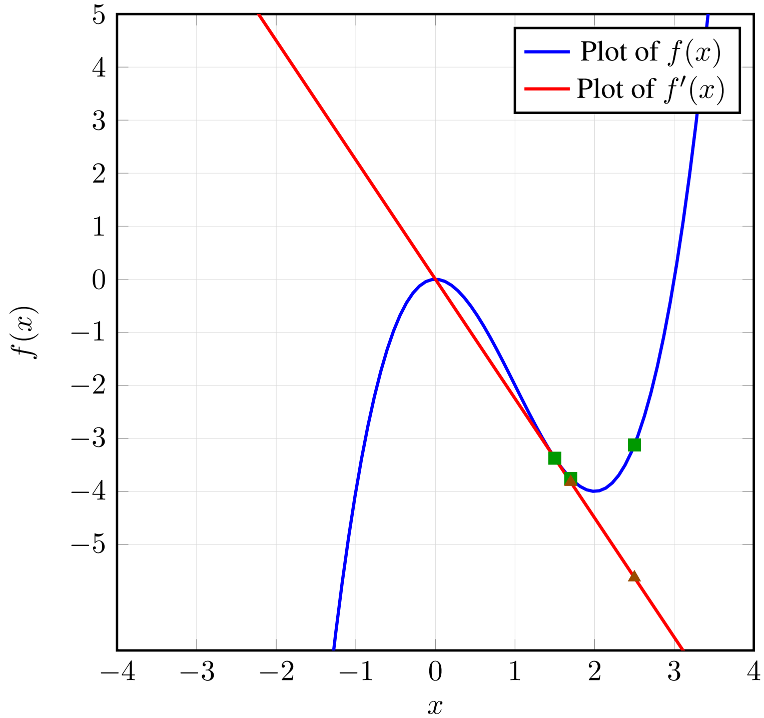
Now that we have this specific point's linearization, we understand how the function behaves around the point. Let's use this linearization to evaluate the function's approximation at $x = 1.7$ (based on our approximation at $x = 1.5$, we want to see how a $\delta = +0.2$ shift in the $x$ value changes the corresponding $f(x)$ value). How does this approximation compare to the exact value of the function at $x = 1.7$?

$$\begin{aligned} f(1.7) &\approx -3.375 + (-2.25) \cdot (1.7 - 1.5) && (2) \\ &\approx -3.375 - 0.45 && (3) \\ &\approx -3.825 && (4) \end{aligned}$$

Comparing to the exact value $f(1.7) = 1.7^3 - 3 \cdot 1.7^2 = -3.757$, we find that the difference is $0.068$. Not too bad! What if we repeat with $\delta = 1$? To do so, we must use the approximation around $x = 1.5$ to compute $x = 2.5$, and compare to the exact value $f(2.5)$. How does our new approximation compare to the exact result?

$$\begin{aligned} f(2.5) &\approx -3.375 + (-2.25) \cdot (2.5 - 1.5) && (5) \\ &\approx -3.375 - 2.25 && (6) \\ &\approx -5.625 && (7) \end{aligned}$$

Comparing to the exact value $f(2.5) = 2.5^3 - 3 \cdot 2.5^2 = -3.125$, we find that the difference is much larger; the error jumped to $2.5$! This is an error multiplication of $\frac{2.5}{0.068} \approx 37$, even though our $\delta$ only multiplied by 5. What happened?
Looking at the actual function, we see that the function has a significant curvature between our "anchor point" of $x_\star = 1.5$ and $x = 2.5$. Our linear model is unable to capture this curvature, and so we estimated $f(2.5)$ as if the function kept decreasing, as it did around $x = 1.5$ (where the slope was $-2.25$).




Now, we can extend this to higher dimensional functions. In the case of a two-dimensional function $f(x,y)$, the linear approximation of $f(x,y)$ at a point $(x_\star, y_\star)$ is given by

$$f(x,y) \approx f(x_\star, y_\star) + f_x(x_\star, y_\star) \cdot (x - x_\star) + f_y(x_\star, y_\star) \cdot (y - y_\star). \tag{8}$$

where $f_x(x_\star, y_\star)$ is the partial derivative of $f(x,y)$ with respect to $x$ at the point $(x_\star, y_\star)$:

$$f_x(x_\star, y_\star) = \left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_\star, y_\star)} \tag{9}$$

and $f_y(x_\star, y_\star)$ is the partial derivative of $f(x,y)$ with respect to $y$ at the point $(x_\star, y_\star)$.

(b) Now, let's see how we can derive partial derivatives. When we are given a function $f(x,y)$, we calculate the partial derivative of $f$ with respect to $x$ by fixing $y$ and taking the derivative with respect to $x$. Given the function $f(x,y) = x^2 y$, find the partial derivatives $f_y(x,y)$ and $f_x(x,y)$.

(c) Write out the linear approximation of $f$ near $(x_\star, y_\star)$.

(d) We want to see if the approximation arising from linearization of this function is reasonable for a point close to our point of evaluation. First, approximate $f(x, y)$ at the point $(2.01, 3.01)$ using $(x_\star, y_\star) = (2, 3)$, and compare the result to $f(2.01, 3.01)$.

(e) Suppose we have now a vector-valued function $f(\vec{x}, \vec{y})$, which takes in vectors $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^k$ and outputs a scalar $\in \mathbb{R}$. That is, $f(\vec{x}, \vec{y})$ is $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$. With this new model, how can we adapt our previous linearization method?

One way to linearize the function $f$ is to do it for every single element in $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top$ and $\vec{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_k \end{bmatrix}^\top$. Then, when we are looking at $x_i$ or $y_j$, we fix everything else as constant. This would give us the linear approximation

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_\star, \vec{y}_\star) + \sum_{i=1}^{n} \frac{\partial f(\vec{x}, \vec{y})}{\partial x_i}(x_i - x_{\star,i}) + \sum_{j=1}^{k} \frac{\partial f(\vec{x}, \vec{y})}{\partial y_j}(y_j - y_{\star,j}). \tag{10}$$

In order to simplify this equation, we can define the rows $D_{\vec{x}}$ and $D_{\vec{y}}$ as

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}, \tag{11}$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \dots & \frac{\partial f}{\partial y_k} \end{bmatrix}. \tag{12}$$

Then, Equation (10) can be rewritten as

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_\star, \vec{y}_\star) + (D_{\vec{x}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star). \tag{13}$$

Assume that $n = k$ and we define the function $f(\vec{x}, \vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x_i y_i$. Find $D_{\vec{x}} f$ and $D_{\vec{y}} f$.

(f) Following the above part, find the linear approximation of $f(\vec{x}, \vec{y})$ near $\vec{x}_\star = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y}_\star = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Recall that $f(\vec{x}, \vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x_i y_i$.

(g) When the function $\vec{f}(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a vector (rather than a scalar), we can view each dimension in $\vec{f}$ independently as a separate function $f_i$, and linearize each of them:

$$\vec{f}(\vec{x},\vec{y}) = \begin{bmatrix} f_1(\vec{x},\vec{y}) \\ f_2(\vec{x},\vec{y}) \\ \vdots \\ f_m(\vec{x},\vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_1 \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_1 \cdot (\vec{y}-\vec{y}_\star) \\ f_2(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_2 \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_2 \cdot (\vec{y}-\vec{y}_\star) \\ \vdots \\ f_m(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_m \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_m \cdot (\vec{y}-\vec{y}_\star) \end{bmatrix} \tag{14}$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} D_{\vec{x}} f_1 \\ D_{\vec{x}} f_2 \\ \vdots \\ D_{\vec{x}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \tag{15}$$

and similarly

$$D_{\vec{y}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \tag{16}$$

Then, the linearization becomes

$$\vec{f}(\vec{x},\vec{y}) \approx \vec{f}(\vec{x}_\star,\vec{y}_\star) + (D_{\vec{x}}\vec{f})\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{x}-\vec{x}_\star) + (D_{\vec{y}}\vec{f})\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{y}-\vec{y}_\star). \tag{17}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$. Find $D_{\vec{x}}\vec{f}$, applying the definition above.

(h) Compare the approximation of $\vec{f}$ at the point $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$ using $\vec{x}_\star = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ versus $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$. Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$.

(i) **Practice:** Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x}, \vec{y}) = \vec{x}\vec{y}^\top \vec{w}$. Find $D_{\vec{x}}\vec{f}$ and $D_{\vec{y}}\vec{f}$.

(j) **Practice:** Continuing the above part, find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

These linearizations are important for us because we can do many easy computations using linear functions.

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