EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet Discussion 8A

For this discussion, Note 8 is helpful.

1. Proctoring Practice

To prepare for the midterm coming up on **March 15, 2021**, let's take a minute to ensure that the proctoring system will work.

Find the email we sent you with a Zoom link (it should have subject line "[EECS 16B] Personal Zoom Proctoring Link for Exams"), join this meeting, and record yourself for five minutes.

In case something went wrong with your Zoom room, fill out this form to tell us what went wrong so we can fix it for you.

2. BIBO Stability

Consider a continuous-time scalar real differential equation with known solution

$$\frac{d}{dt}x(t) = ax(t) + bu(t) \qquad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\, d\tau.$$

Show that if the system has $\text{Re}\{a\} > 0$, then a bounded input can result in an unbounded output (i.e., the system is unstable) for every initial condition $x(0)$.

$|u(t)| \leq \varepsilon \quad \forall t$

let $u(t) = \varepsilon$

$$\int_0^t e^{a(t-\tau)} bu(\tau) d\tau = be^{at} \int_0^t e^{-a\tau} \cdot \varepsilon\, d\tau = b\varepsilon e^{at} \cdot \frac{-1}{a} e^{a\tau}\Big|_0^t$$

$$= \frac{b}{a} \varepsilon e^{at}(1 - e^{-at}) = \frac{b}{a}\varepsilon(e^{at} - 1)$$

$a = Re\{a\} + jIm\{a\}$

$$= \frac{b}{a}\varepsilon\Big(\underbrace{e^{Re\{a\}\cdot t}}_{\to\infty} e^{jIm\{a\}t} - 1\Big)$$

$$x(t) = \underbrace{e^{at}}_{\to\infty} x(0) + \frac{b}{a}\varepsilon(\underbrace{e^{at}}_{\to\infty} - 1)$$

$\therefore\ x(t)$ is unbounded

3. Changing behavior through feedback

In this question, we discuss how feedback control can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[t+1] = 0.9x[t] + u[t] + w[t] \tag{1}$$

where $u[t]$ is the control input we get to apply based on the current state and $w[t]$ is the external disturbance, each at time t.

Is the system stable? If $|w[t]| \leq \varepsilon$, what can you say about $|x[t]|$ at all times t if you further assume that $u[t] = 0$ and the initial condition $x[0] = 0$? How big can $|x[t]|$ get?

$|0.9| < 1 \Rightarrow$ Stable

$$x[t] = \sum_{k=0}^{t-1} 0.9^{t-k-1} w[k]$$

$$|x[t]| = \left| \sum_{k=0}^{t-1} 0.9^{t-k-1} w[k] \right| \leq \sum_{k=0}^{t-1} 0.9^{t-k-1} |w[k]| \leq \varepsilon \sum_{k=0}^{t-1} 0.9^{t-k-1}$$

$$= \varepsilon \cdot \frac{1}{1-0.9} = 10\varepsilon$$

in limit, $|x[t]| \leq 10\varepsilon$

(b) Suppose that we decide to choose a control law $u[t] = kx[t]$ to apply in feedback. For what values of λ can you get the system to behave like:

$$x[t+1] = \lambda x[t] + w[t]? \tag{2}$$

How would you pick k?

(*Note*: In this case, $w[t]$ can be thought of like another input to the system, except we can't control it.)

(*Note*: In lecture we call this term f – for feedback – instead of k, but we use k here since it's a more traditional notation for feedback, and also lowercase f is confused with functions.)

$$x[t+1] = 0.9x[t] + kx[t] + w[t] = (0.9+k)x[t] + w[t]$$

$$\lambda = 0.9 + k$$

$$k = \lambda - 0.9$$

(c) For the previous part, which k would you choose to minimize how big $|x[t]|$ can get?

$|w[t]| \leq \varepsilon \qquad x[t+1] = \lambda x[t] + w[t]$

$$\lambda = 0$$

$$k = -0.9$$

(d) What if instead of a 0.9, we had a 3 in the original eq. (1). What, if anything, would change?

$x[t+1] = 3x[t] + u[t] + w[t]$

$|3| > 1 \Rightarrow$ unstable

$K = \lambda - 3$

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] + \vec{w}[t] \tag{3}$$

where we further assume that B is an invertible square matrix.
Suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[t] = F\vec{x}[t]$. For what values of matrix G can you get the system to behave like:

$$\vec{x}[t+1] = G\vec{x}[t] + \vec{w}[t]? \tag{4}$$

How would you pick F given knowledge of A, B and the desired goal dynamics G?

$x[t+1] = Ax[t] + BFx[t] + w[t]$

$= (A + BF)x[t] + w[t]$

$A + BF = G$

$F = B^{-1}(G - A)$

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