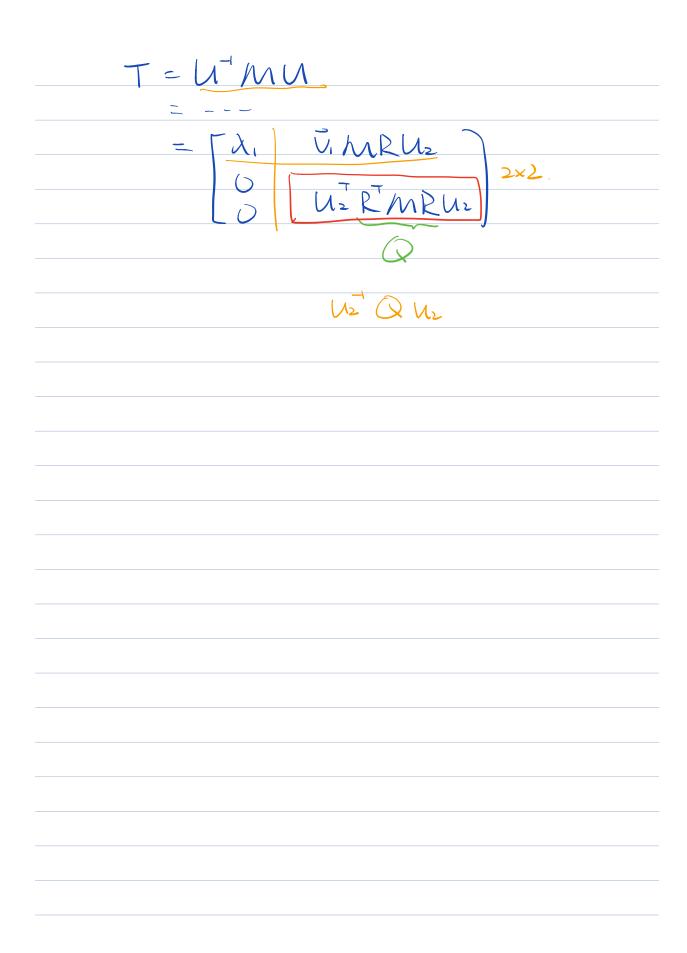
EECS 16B DIS 10B Casque
Upper-triangularization
M: any sprave matrix nxn
want: M=UTUT
U: Orthogoner matrix
T: upper triangular morrix
2x2 case: M= [M11 M12] [M21 M21]
m21 M22
Let Vi be an eigenvector of M with nom=
Let Vi be an eigenvector of M with norm=
Let $y = [\vec{v}, R]$
2x2 from Civenn-Schmidt
=> T= U ^T MU
$= \begin{bmatrix} \lambda, & \overline{V},^{T}MR \\ O & R^{T}MR \end{bmatrix}$
O RTMR)
upper-tranquiar!

Let $U = [V_1 \ R]$ from Cwam-Schmidt 3×3 3×1 3×2

$$(2\times3)(3\times2)=2\times2$$

Based on the 2x2 case, we know $\exists M_2 \text{ sit } (N_2)^{-1} Q N_2$ $= (N_1)^{-1} P^T M P N_2$ $= (N_2)^{-1} P^T M P N_2$

is upper-triangular



1. Towards upper-triangulation by an orthonormal basis

This problem is a continuation of problem 1 from Discussion 10A.

Recall that in the last discussion we set out to show that any matrix M can be upper triangularized. In particular we want to find the coordinate transformation U such that M becomes upper triangular when represented in this coordinate system:

$$T = U^{-1}MU. (1)$$

In the previous discussion we began with the example of a 3x3 matrix M. We first constructed U by extending the first eigenvector of M, \vec{v}_1 , into an orthonormal basis using Gram-Schmidt:

$$\underline{U} = \begin{bmatrix} \vec{v}_1 & R \\ & & \\ & & \\ \end{bmatrix}.$$
(2)

This gave us the transformed matrix:

$$T = \begin{bmatrix} \lambda_1 & \vec{a}^{\mathsf{T}} \\ \vec{0} & Q \end{bmatrix} \qquad 2 \times 2 \tag{3}$$

which is upper triangular if Q is upper triangular. Then we realized that we could do a similar transformation on Q to get

$$Q = \begin{bmatrix} \vec{v}_2 & \vec{V} \end{bmatrix} \begin{bmatrix} \lambda_2 & \vec{b}^{\mathsf{T}} \\ \vec{0} & P \end{bmatrix} \begin{bmatrix} \vec{v}_2 & \vec{Y} \end{bmatrix}^{\mathsf{T}}$$

$$(4)$$

where \vec{v}_2 is the first eigenvector of matrix Q and corresponds to λ_2 . At this point, since Q was a 2x2 matrix, we can see that the transformed Q is upper triangular. We then plugged this Q in for M, and simplified to get the final result:

$$M = \begin{bmatrix} \vec{v}_1 & \underbrace{\vec{R}\vec{v}_2 & \vec{R}\vec{Y}} \\ \boxed{\hat{V}_1} & \underbrace{\vec{k}_1} & \check{a}_1 & \check{a}_{rest} \\ 0 & \lambda_2 & \vec{b} \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} \vec{v}_1 & R\vec{v}_2 & RY \end{bmatrix}^{\top}.$$
 (5)

(a) Show that the matrix $\begin{bmatrix} \vec{v}_1 & R\vec{v}_2 & RY \end{bmatrix}$ is still orthonormal.

$$V_1$$
 and $P_1 = V_2 = V_1 = V_2 = V_2 = V_3 = V_4 =$

$$(R\vec{V}_2)^T R\vec{V}_2 = \vec{V}_2^T R^T R\vec{V}_2$$
$$= \vec{V}_2^T \vec{V}_2 = \vec{V}_2^T \vec{V}_2 =$$

$$(RY)^{T}RY = Y^{T}R^{T}RY$$

$$= Y^{T}Y = 1$$

(b) We have shown how to upper triangularize a 3×3 and a 2×2 matrix. How can we generalize this process to an nxn matrix?

Amy
$$M_i \in \mathbb{R}^{n \times n}$$

$$M_i = \begin{bmatrix} \vec{v}_i & R_i \end{bmatrix} \begin{bmatrix} \lambda_i & \vec{a}_i^T \\ \vec{o} & M_2 \end{bmatrix} \begin{bmatrix} \vec{v}_i^T \\ \vec{P}_i^T \end{bmatrix}$$

$$(n-1) \times (n-1)$$

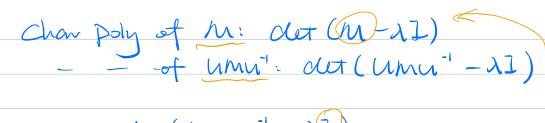
$$M_2 = \begin{bmatrix} \vec{v}_2 & \vec{P}_2 \end{bmatrix} \begin{bmatrix} \vec{\lambda}_1 & \vec{\alpha}_2 \end{bmatrix} \begin{bmatrix} \vec{V}_2^T \\ \vec{O} & M_3 \end{bmatrix} \begin{bmatrix} \vec{V}_2^T \\ \vec{P}_2^T \end{bmatrix}$$

Corrime until M: ER2XL

motil we have U.

$$M_i = U_i T U_i^T$$
 $T = U_i^T M_i U_i$

(c) Show that the characteristic polynomial of square matrix M is the same as that of the square matrix UMU^{-1} for any invertible U.



$$det(u) \cdot det(u^{-1}) = 1$$

$$= det(h - \lambda 1)$$

2. Minimum Energy Control

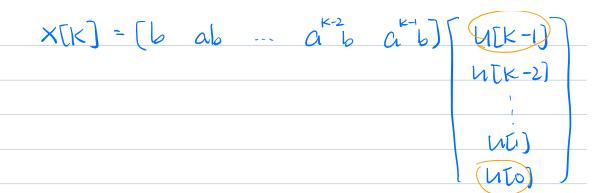
In this question, we build up an understanding for how to get the minimum energy control signal to go from one state to another

(a) Consider the scalar system:

$$x[k+1] = ax[k] + bu[k] \tag{21}$$

where x[0] = 0 is the initial condition and u[k] is the control input we get to apply based on the current state. Consider if we want to reach a certain state, at a certain time, namely x[K]. Write a matrix equation for how a choice of values of u[k] for $k \in \{0, 1, \dots K-1\}$ will determine the output at time K.

[Hint: write out all the inputs as a vector $[u[0] \ u[1] \ \cdots \ u[K-2] \ u[K-1]]^T$ and figure out the combination of a and b that gives you the state at time K.]



$$x[k+1] = 1.0x[k] + 0.7u[k]$$
 (22)

where x[0] = 0 is the initial condition and u[k] is the control input we get to apply based on the current state. Suppose if we want to reach a certain state, at a certain time, namely x[K] = 14. With our dynamics a = 1, solve for the best way to get to a specific state x[K] = 14, when K = 10. When we say **best way** to control a system, we want the sum squared of the inputs to be minimized

$$\underset{u[k]}{\operatorname{argmin}} \sum_{k=0}^{K} u[k]^{2}.$$

Hint: recall the Cauchy-Schwarz inequality $\langle \vec{a}, \vec{b} \rangle \leq \|\vec{a}\| \|\vec{b}\|$ where equality holds if \vec{a} and \vec{b} are linearly dependent.

< v. V > = ||v| || vhen v. V theny dependent



ū = 2

(c) Consider the scalar system:

$$x[t+1] = 0.5x[t] + 0.7u[t]$$
 (23)

where x[0] = 0 is the initial condition and u[t] is the control input we get to apply based on the current state. Consider if we want to reach a certain state, at a certain time, namely x[K] = 14, when K = 10. Explain in words the trend of the control input that will be used to solve this problem.

$$X[K] = [0.] 0.7 \cdot 0.5 - ... 0.7 \cdot 0.5^{K-1}] [u[K-1]]$$
 $[u[V] = [0.] 0.7 \cdot 0.5 - ... 0.7 \cdot 0.5^{K-1}] [u[V] = [0.] 0.7 \cdot 0.7 \cdot 0.5^{K-1}] [u[V] = [0.] 0.7 \cdot 0.$

$$X[10] = 14 = \langle \vec{v}, \vec{u} \rangle$$

= $\times \langle \vec{v}, \vec{v} \rangle$