

EECS 16B DRS 5B Crazyue

Phasors!

-  $\sin(x) = \cos(x - \frac{\pi}{2})$

- Euler's Identity

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

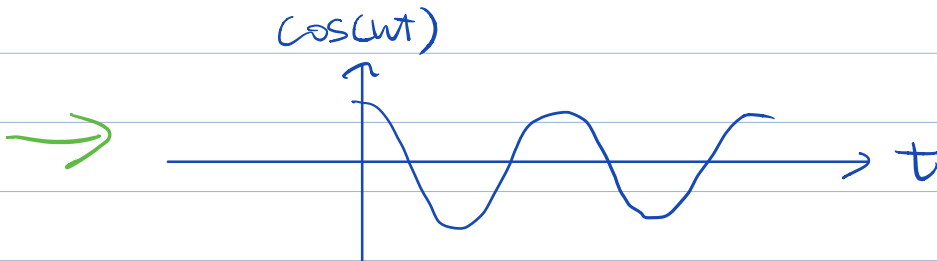
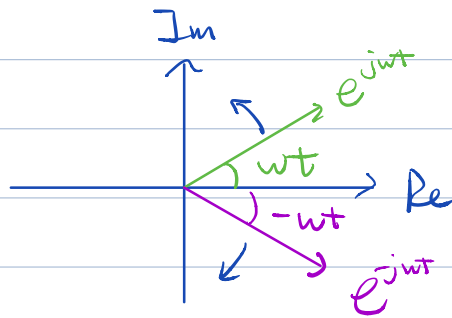
$$\Rightarrow e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta)$$

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Similarly,  $\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$



- Any sinusoid,

$$v(t) = V_0 \cos(\omega t + \phi)$$

$$= \frac{V_0}{2} (e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)})$$

$$\tilde{V} = \frac{1}{2} V_0 e^{-j\phi}$$
$$= \frac{V_0}{2} e^{j\phi} e^{j\omega t} + \frac{V_0}{2} e^{-j\phi} e^{-j\omega t}$$

$\tilde{V}$                        $\tilde{V}$

define this to be the phasor.

- Impedance

$$Z = \frac{\tilde{V}}{\tilde{I}}$$

Resistance:  $Z_R = R$

Capacitance:  $Z_C = \frac{1}{j\omega C}$

Inductance:  $Z_L = j\omega L$

## 1. Phasor Analysis

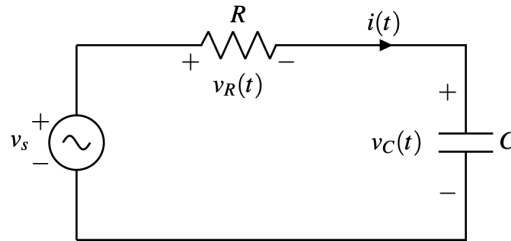
Any sinusoidal time-varying function  $x(t)$ , representing a voltage or a current, can be expressed in the form

$$x(t) = \tilde{X}e^{j\omega t} + \bar{\tilde{X}}e^{-j\omega t}, \quad (1)$$

where  $\tilde{X}$  is a time-independent function called the **phasor** representation of  $x(t)$  (recall that  $\bar{a}$  denotes the complex conjugate of  $a$ ). Note that 1)  $\tilde{X}$  and  $\bar{\tilde{X}}$  are complex conjugates of each other, 2)  $e^{j\omega t}$  and  $e^{-j\omega t}$  are complex conjugates of each other, and 3) that  $\tilde{X}e^{j\omega t}$  and  $\bar{\tilde{X}}e^{-j\omega t}$  are also complex conjugates of each other.

**Note:** We define the phasor as the coefficient of  $e^{j\omega t}$  in eq. (1). Other resources (such as some past iterations of this class) define it slightly differently; the definitions differ by a constant multiple. Some reasons for competing definitions are discussed in **Note 4**. Although the definitions in general lead to the same answers, be careful to use our class' definition and not get tripped up. For example, if we ask about the magnitude of the phasor, you wouldn't want to be off by a constant!

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s(t) = 12 \sin\left(\omega t - \frac{\pi}{4}\right), \quad (2)$$

with  $\omega = 1 \times 10^3 \frac{\text{rad}}{\text{s}}$ ,  $R = \sqrt{3}\text{k}\Omega$ , and  $C = 1 \mu\text{F}$ .

Our goal is to obtain a solution for  $i(t)$  with the sinusoidal voltage source  $v_s(t)$ .

(a) **Step 1: Write sources as exponentials:**  $\tilde{X}e^{j\omega t} + \bar{\tilde{X}}e^{-j\omega t}$

All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. Convert  $v_s(t)$  into an exponential and write down its phasor representation  $\tilde{V}_s$ .

$$\begin{aligned} v_s(t) &= 12 \sin\left(\omega t - \frac{\pi}{4}\right) \\ &= 12 \cos\left(\omega t - \frac{3\pi}{4}\right) \\ &= 12 \left[ \frac{1}{2} \left( e^{j\left(\omega t - \frac{3\pi}{4}\right)} + e^{-j\left(\omega t - \frac{3\pi}{4}\right)} \right) \right] \\ &= \underbrace{6}_{\tilde{V}_s} e^{j\frac{3\pi}{4}} e^{j\omega t} + \underbrace{6}_{\bar{\tilde{V}_s}} e^{j\frac{3\pi}{4}} e^{-j\omega t} \end{aligned}$$

$V_s$  $V_s$ 

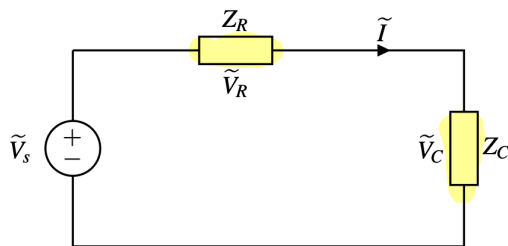
$$\tilde{V}_s = 6e^{-j\frac{3\pi}{4}}$$

(b) **Step 2: Transform circuits to phasor domain**

The voltage source is represented by its phasor  $\tilde{V}_s$ . The current  $i(t)$  is related to its phasor counterpart  $\tilde{I}$  by

$$i(t) = \tilde{I}e^{j\omega t} + \tilde{I}^*e^{-j\omega t}.$$

We redraw the circuit in phasor domain:



What are the impedances of the resistor,  $Z_R$ , and capacitor,  $Z_C$ ? We sometimes also refer to this as the "phasor representation" of  $R$  and  $C$ .

$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C}$$

(c) **Step 3: Cast the branch and element equations in phasor domain**

Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

$$\rightarrow V_s(t) = V_R(t) + V_C(t)$$

$$\tilde{V}_s e^{j\omega t} + \tilde{V}_s e^{-j\omega t} = \tilde{V}_R e^{j\omega t} + \tilde{V}_R e^{-j\omega t} + \tilde{V}_C e^{j\omega t} + \tilde{V}_C e^{-j\omega t}$$

$$\rightarrow (\tilde{V}_s - \tilde{V}_R - \tilde{V}_C) e^{j\omega t} + (\tilde{V}_s - \tilde{V}_R - \tilde{V}_C) e^{-j\omega t} = 0$$

= 0

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$$\tilde{V}_s - \tilde{V}_R - \tilde{V}_C = 0$$

$$\tilde{V}_s = \tilde{V}_R + \tilde{V}_C$$

⇒ KVL holds for phasors

$$\tilde{V}_s = \tilde{V}_R + \tilde{V}_C$$

$$Z = \frac{\tilde{V}}{\tilde{I}}$$

$$6e^{-j\frac{3\pi}{4}} = R\tilde{I} + \frac{1}{j\omega C}\tilde{I}$$

$Z_R$

$Z_C$

(d) Step 4: Solve for unknown variables

Solve the equation you derived in Step 3 for  $\tilde{I}$  and  $\tilde{V}_C$ . What is the polar form of  $\tilde{I}$  and  $\tilde{V}_C$ ? Polar form is given by  $Ae^{j\theta}$ , where  $A$  is a positive real number.

$$\tilde{I} = \frac{6e^{-j\frac{3\pi}{4}}}{R + \frac{1}{j\omega C}} \cdot \frac{j\omega C}{j\omega C}$$

$$\Rightarrow = \frac{j\omega C e^{j\frac{3\pi}{4}}}{j\omega RC + 1} \tilde{I}$$

$$\tilde{V}_C = \tilde{I} Z_C = \frac{j\omega C e^{-j\frac{3\pi}{4}}}{j\omega RC + 1} \cdot \frac{1}{j\omega C}$$
$$= \frac{6e^{j\frac{3\pi}{4}}}{j\omega RC + 1}$$

$$j e^{j\dots}$$
$$e^{j\frac{\pi}{2}}$$
$$e^{j\dots}$$

Polar Form:  $Ae^{j\theta}$

$$\frac{A}{(1+j)} \frac{1-j}{(1-j)}$$

Plug in values ----

$$\omega = 10^3 \frac{\text{rad}}{\text{s}}$$

$$R = \sqrt{3} \times 10^3 \Omega$$

$$C = 1 \times 10^{-6} \text{F}$$

$$\Rightarrow \tilde{I} = 3e^{-j\frac{7\pi}{12}} \text{ mA}$$

$$\tilde{V}_c = 3e^{-j\frac{13\pi}{12}} \text{ V}$$

$$\frac{1}{j} = -j$$

(e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is  $i(t)$  and  $v_c(t)$ ? What is the phase difference between  $i(t)$  and  $v_c(t)$ ?

$$i(t) = \tilde{I} e^{j\omega t} + \tilde{I} e^{-j\omega t}$$

$$= 3e^{-j\frac{7\pi}{12}} e^{j\omega t} + 3e^{j\frac{7\pi}{12}} e^{-j\omega t}$$

$$= 6 \cos\left(\omega t - \frac{7\pi}{12}\right) \text{ mA}$$

$$v_c(t) = \tilde{V}_c e^{j\omega t} + \tilde{V}_c e^{-j\omega t}$$

$$= 3e^{-j\frac{13\pi}{12}} e^{j\omega t} + 3e^{j\frac{13\pi}{12}} e^{-j\omega t}$$

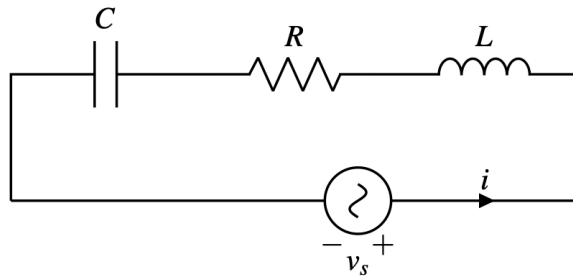
$$= 6 \cos\left(\omega t - \frac{13\pi}{12}\right) \text{ V}$$

$$\angle \tilde{V}_c - \angle \tilde{I} = -\frac{13\pi}{12} - \left(-\frac{7\pi}{12}\right) = -\frac{\pi}{2}$$

## 2. RLC Circuit Phasor Analysis

We study a simple RLC circuit with an AC voltage source given by

$$v_s(t) = B \cos(\omega t - \phi)$$



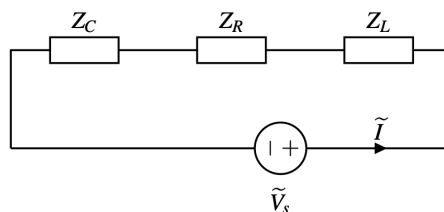
(a) Write out the phasor representation of  $v_s(t)$ , and the impedances of  $R$ ,  $C$ , and  $L$ .

$$v_s(t) = B \cos(\omega t - \phi)$$

$$\begin{aligned} \rightarrow &= \frac{B}{2} \left( e^{j(\omega t - \phi)} + e^{-j(\omega t - \phi)} \right) \\ &= \frac{B}{2} e^{-j\phi} e^{j\omega t} + \frac{B}{2} e^{j\phi} e^{-j\omega t} \\ &\quad \underbrace{\hspace{1.5cm}}_{\tilde{V}_s} \end{aligned}$$

$$\left. \begin{aligned} Z_R &= R \\ Z_C &= \frac{1}{j\omega C} \\ Z_L &= j\omega L \end{aligned} \right\}$$

(b) Now, we're going to redraw the circuit in the phasor domain.



Use Kirchhoff's laws to write down a loop equation relating the phasors.

KVL:

$$Z_R \tilde{I} + Z_C \tilde{I} + Z_L \tilde{I} = \tilde{V}_s$$

$$(R + \frac{1}{j\omega C} + j\omega L) \tilde{I} = \frac{B}{2} e^{-j\phi}$$

(c) Solve the equation in the previous step for the current  $\tilde{I}$ . What is the magnitude and phase of the polar form of  $\tilde{I}$ ?

Hint: You'll need the following identities, which you can find in Note j:

- $|z_1/z_2| = |z_1|/|z_2|$
- $\angle(\frac{z_1}{z_2}) = \angle z_1 - \angle z_2$
- $\angle(a + jb) = \text{atan2}(b, a)$

$$\tilde{I} = \frac{\frac{B}{2} e^{-j\phi}}{R + \frac{1}{j\omega C} + j\omega L}$$

$$= \frac{\frac{B}{2} e^{-j\phi}}{R + j(\omega L - \frac{1}{\omega C})}$$

$$A e^{j\phi}$$

$$|A| \quad \phi$$

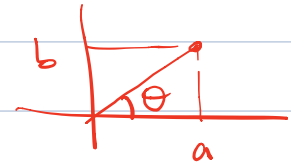
$$\frac{1}{j} = -j$$

Magnitude:

$$|\tilde{I}| = \frac{| \dots |}{| \dots |}$$

$$= \frac{\frac{B}{2}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$$a + jb$$



$$\theta = \text{atan2}(b, a)$$

Phase:

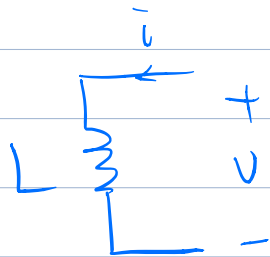
$$\angle \tilde{I} = \angle \text{num} - \angle \text{denom}$$

$$= -\phi - \text{atan2}(\omega L - \frac{1}{\omega C}, R)$$



### 3. (Practice) Inductor Impedance

Given the voltage-current relationship of an inductor  $v(t) = L \frac{di(t)}{dt}$ , show that its complex impedance is  $Z_L = j\omega L$ .



$$\text{Assume } i(t) = I_0 \cos(\omega t + \phi)$$

$$= \frac{I_0}{2} e^{j\phi} e^{j\omega t} + \frac{I_0}{2} e^{-j\phi} e^{-j\omega t}$$

$\underbrace{\hspace{1.5cm}}_{\tilde{I}}$

$$v(t) = L \frac{di(t)}{dt}$$

$$= L \frac{d}{dt} ( \quad )$$

$$= \frac{j\omega L I_0 e^{j\phi}}{2} e^{j\omega t} + \frac{-j\omega L I_0 e^{-j\phi}}{2} e^{-j\omega t}$$

$$= \underbrace{j\omega L \tilde{I}}_{\tilde{V}} e^{j\omega t} + \underbrace{j\omega L \tilde{I}}_{\tilde{V}} e^{-j\omega t}$$

$$\Rightarrow \underbrace{\tilde{V}}_{Z_L} = j\omega L \tilde{I}$$

$$j = e^{j\frac{\pi}{2}}$$

