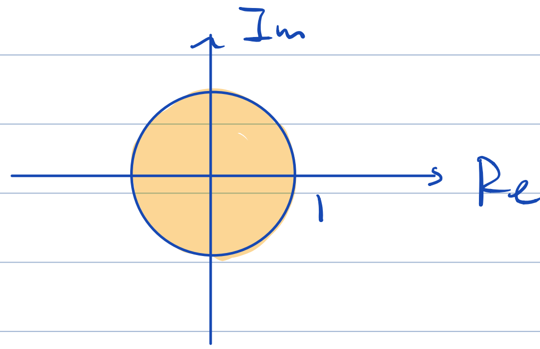
EECS 16B DIS 8B Gaoyue

Stability

Discrete-Time: $\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$

$|\lambda_i(A)| < 1 \quad \forall i$



Controllability

$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$

$\rightarrow \vec{x}[1] = A\vec{x}[0] + B\vec{u}[0]$

$\rightarrow \vec{x}[2] = A\vec{x}[1] + B\vec{u}[1]$

$= A(A\vec{x}[0] + B\vec{u}[0]) + B\vec{u}[1]$

$= A^2\vec{x}[0] + AB\vec{u}[0] + B\vec{u}[1]$

$\vec{x} \in \mathbb{R}^n$

$= A^2\vec{x}[0] + [AB \quad B]\begin{bmatrix}\vec{u}[0] \\ \vec{u}[1]\end{bmatrix}$

$\vec{x}[t] = A^n \vec{x}[0] + [B \ AB \ \cdots \ A^{t-1}B] \begin{bmatrix} \vec{u}[t-1] \\ \vdots \\ \vec{u}[0] \end{bmatrix}$

n

$C = [B \ AB \ \cdots \ A^{n-1}B] \ A^nB \ A^{n+1}B$

System is controllable if column span of C is n-dimensional

↑ state dimension

1. Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] + \vec{w}[t] \tag{1}$$

(a) Is the system given in eq. (1) stable?

$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Need eigenvalues of A

$\det(A - \lambda I) = 0$

$(-\lambda)(-1-\lambda) - 2 = 0$

$\lambda + \lambda^2 - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$

$\lambda = 1, -2$

$|\lambda| > 1 \Rightarrow$ unstable.

(b) Derive a state space representation of the resulting closed loop system using state feedback of the form $u[t] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t]$.

Hint: If you're having trouble parsing this expression for $u[t]$, note that $\begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is a *row vector*, while $\vec{x}[t]$ is a *column vector*. What happens when we multiply a row vector with a column vector like this?

$$\vec{x}[t+1] = A\vec{x}[t] + \vec{b}\,u[t]$$

$$= A\vec{x}[t] + \vec{b}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\vec{x}[t]$$

$$= \left(A + \vec{b}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\vec{x}[t]$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\vec{x}[t]$$

$$= \begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}\vec{x}[t]$$

(c) Find the appropriate state feedback constants, k_1, k_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

$$\vec{x}[t+1] = \underbrace{\begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}}_{A_{cl}}\vec{x}[t]$$

eigenvalues of A_{cl}:

$$\det(A_{cl} - \lambda I) = 0$$

$$(k_1-\lambda)(-1-\lambda)-2(1+k_2)=0$$

$$-k_1-k_1\lambda+\lambda+\lambda^2-2-2k_2=0$$

(1) $$\lambda^2+(1-k_1)\lambda+(-k_1-2k_2-2)=0$$

need $\lambda_1=-\frac{1}{2}$, $\lambda_2=\frac{1}{2}$

$$\Rightarrow (\lambda+\tfrac{1}{2})(\lambda-\tfrac{1}{2})=0$$

(2) $$\lambda^2-\tfrac{1}{4}=0$$

$$\rightarrow \begin{cases} 1-k_1=0 \\ -\frac{1}{4}=-k_1-2k_2-2 \end{cases}$$

$$\Rightarrow \begin{cases} k_1=1 \\ k_2=-\frac{11}{8} \end{cases}$$

(d) Is the system now stable?

Yes!

(e) Suppose that instead of $\begin{bmatrix}1\\0\end{bmatrix}u[t]$ ($\vec{b}$) in eq. (1), we had $\begin{bmatrix}1\\1\end{bmatrix}u[t]$ (new $\vec{b}$) as the way that the discrete-time control acted on the system. As before, we use $u[t]=\begin{bmatrix}k_1 & k_2\end{bmatrix}\vec{x}[t]$ to try and control the system. What would the eigenvalues be? Can you move all the eigenvalues to where you want? Give an intuitive explanation of what is going on.

$$\vec{x}[t+1]=\left(A+\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}\right)\vec{x}[t]$$

(new $\vec{b}$)

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix} \right) \vec{x}[t]$$

$$= \begin{bmatrix} k_1 & k_2+1 \\ k_1+2 & k_2-1 \end{bmatrix} \vec{x}[t]$$

$$\det(\quad - \lambda I) = 0$$

$$(\lambda+2)(\lambda - (1+k_1+k_2)) = 0$$

-2

always have an $\lambda = -2$

Cannot arbitrarily change eigenvalues

2. Controlling states by designing sequences of inputs

This is something that you saw in 16A in the Segway problem. In that problem, you were given a semi-realistic model for a segway. Here, we are just going to consider the following matrix chosen for ease of understanding what is going on:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[t+1] = A\vec{x}[t] + \vec{b}u[t].$$

(a) We are given the initial condition $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}[T] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $T \geq 0$. We don't need to stay there, we just want to be in this state at that time. What is the smallest T such that this is possible? What is our choice of sequence of inputs $u[t]$?

$$\vec{x}[t] = \begin{bmatrix} x_1[t] \\ x_2[t] \\ x_3[t] \\ x_4[t] \end{bmatrix}$$

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix}$$

$$\vec{x}[2] = \begin{bmatrix} x_3[0] \\ x_4[0] \\ u[0] \\ u[1] \end{bmatrix} \quad \vec{x}[3] = \begin{bmatrix} x_4[0] \\ u[0] \\ u[1] \\ u[2] \end{bmatrix} \quad \vec{x}[t] = \begin{bmatrix} u[t-4] \\ u[t-3] \\ u[t-2] \\ u[t-1] \end{bmatrix}$$

(b) What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest T and what is our choice of $u[t]$?

(c) What if we started from $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$? What is the smallest T and what is our choice of $u[t]$?

3. Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}[t+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[t]$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) Is the system controllable?

(b) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T - 1$?

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0]$$

$$= \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}$$

$$\vec{x}[2] = \begin{bmatrix} 4 \\ -6+2u[0] \\ -3+2u[1] \end{bmatrix}$$

$$\vec{x}[3] = \begin{bmatrix} 8 \\ -15+2u[1] \\ -6+2u[0]+2u[1] \end{bmatrix}$$

$$\vec{x}[t] = \begin{bmatrix} 2^t x_1[0] \\ -3x_1[t-1] + x_3[t-1] \\ x_2[t-1] + 2u[t-1] \end{bmatrix}$$

(c) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T - 1$?

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}$$

let $u = -1$

(d) Find the set of all possible states reachable after two timesteps.

$$\vec{x}[2] = \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$