EECS 16B DIS 9B Gaoyue

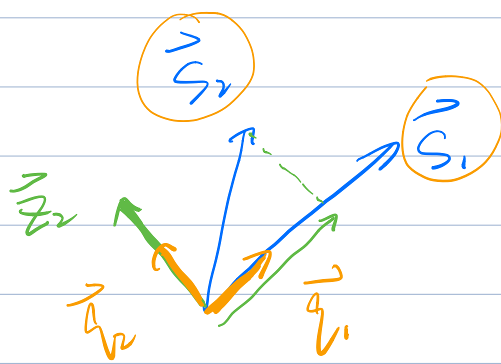
Gram Schmidt

a set of linearly independent vectors

$\{\vec{s}_1, \vec{s}_2, ---, \vec{s}_k\}$

orthonormal

want: $\{\vec{q}_1, --- \vec{q}_k\}$ that spans the same k-dimensional subspace as $\{\vec{s}_1, ---, \vec{s}_k\}$

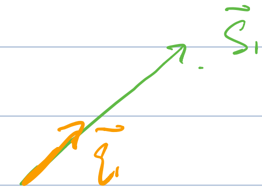


1. Gram-Schmidt Algorithm

Let's apply Gram-Schmidt orthonormalization to a set of three linearly independent vectors $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$.

(a) Find unit vector $\vec{q}_1$ such that $\text{span}(\{\vec{q}_1\}) = \text{span}(\{\vec{s}_1\})$.

$$\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|}$$



(b) Given $\vec{q}_1$ from the previous step, find $\vec{q}_2$ such that $\text{span}(\{\vec{q}_1, \vec{q}_2\}) = \text{span}(\{\vec{s}_1, \vec{s}_2\})$ and $\vec{q}_2$ is orthogonal to $\vec{q}_1$.
What would happen if $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ were *not* linearly independent, but rather $\vec{s}_1$ were a multiple of $\vec{s}_2$?

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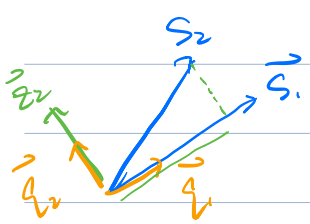
$$\vec{z}_2 = \vec{s}_2 - (\vec{s}_2^{\,T} \vec{q}_1)\, \vec{q}_1$$

normalize $\vec{z}_2$:

$$\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|}$$



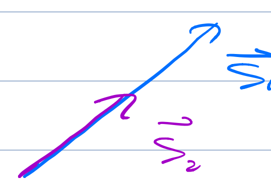
$\rightarrow \langle \vec{s}_2, \vec{q}_1 \rangle$

$\rightarrow = \vec{s}_2^{\,T} \vec{q}_1$

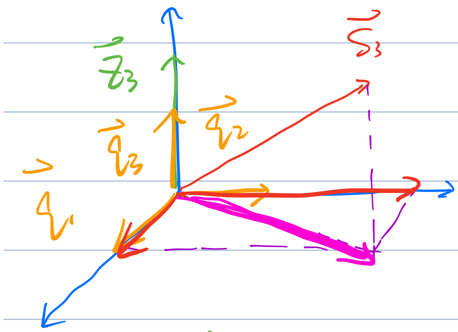
when $\vec{s}_2 = \alpha \vec{s}_1$

$$\vec{z}_2 = 0$$

$$\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\| \leftarrow 0}$$



(c) Now given $\vec{q}_1$ and $\vec{q}_2$ in the previous steps, find $\vec{q}_3$ such that $\text{span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$, and $\vec{q}_3$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, and finally $\|\vec{q}_3\| = 1$.



$$\vec{z}_3 = \vec{s}_3 - \underbrace{(\vec{s}_3^{\,T}\vec{q}_1)\vec{q}_1 - (\vec{s}_3^{\,T}\vec{q}_2)\vec{q}_2}_{\text{Proj}_{\text{span}\{q_1, q_2\}} \vec{s}_3}$$

$$\vec{q}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|}$$

(d) Let's extend this algorithm to n linearly independent vectors. That is, given an input $\{\vec{s}_1, \ldots, \vec{s}_n\}$, write the algorithm to calculate the orthonormal set of vectors $\{\vec{q}_1, \ldots, \vec{q}_n\}$, where $\text{span}(\{\vec{s}_1, \ldots, \vec{s}_n\}) = \text{span}(\{\vec{q}_1, \ldots, \vec{q}_n\})$.

Hint: How would you calculate the i^{th} vector, $\vec{q}_i$?

Step 1: $\vec{q}_1 = \dfrac{\vec{s}_1}{\|\vec{s}_1\|}$

for $i = 2, \ldots, n$:

$$\vec{z}_i = \vec{s}_i - \sum_{j=1}^{i-1} (\vec{s}_i^{\,T}\vec{q}_j)\vec{q}_j$$

normalize $\vec{z}_i$ to get $\vec{q}_i$

$$\vec{q}_i = \frac{\vec{z}_i}{\|\vec{z}_i\|}$$

2. The Order of Gram-Schmidt

If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the set of vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Perform Gram-Schmidt on these vectors first in the order $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{q}_2 = \vec{v}_2 - \underbrace{(\vec{v}_2^T \vec{q}_1)}_{=1} \vec{q}_1 = \vec{v}_2 - \vec{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{q}_3 = \vec{v}_3 - \underline{(\vec{v}_3^T \vec{q}_1)\vec{q}_1} - \underline{(\vec{v}_3^T \vec{q}_2)\vec{q}_2}$$

$$= \vec{v}_3 - \vec{q}_1 - \vec{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Now perform Gram-Schmidt on these vectors in the order $\vec{v}_3, \vec{v}_2, \vec{v}_1$. Do you get the same result?

$$\vec{q}_1 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}$$

$$\|\vec{v}_3\| = \left\|\begin{bmatrix}1\\1\\1\end{bmatrix}\right\| = \sqrt{3}$$

$$\vec{z}_2 = \vec{v}_2 - (\vec{v}_2^T \vec{q}_1)\,\vec{q}_1$$

$$= \vec{v}_2 - \frac{2}{\sqrt{3}}\vec{q}_1 = \begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ -\frac{2}{3}\end{bmatrix}$$

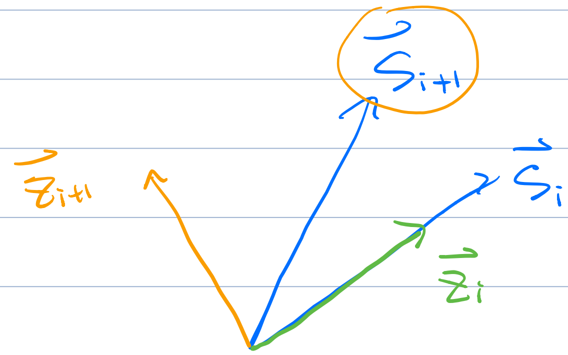
$$\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} = \frac{1}{\sqrt{2/3}}\vec{z}_2 = \begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ -\sqrt{\frac{2}{3}}\end{bmatrix}$$

$$\vec{z}_3 = \vec{v}_3 - (\vec{v}_3^T \vec{q}_1)\,\vec{q}_1 - (\vec{v}_3^T \vec{q}_2)\,\vec{q}_2$$

$$= \begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ 0\end{bmatrix}$$

$$\vec{q}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}$$

$$\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}.$$



$$\vec{z}_i = \vec{v}_i - (\vec{v}_i^{\,T}\vec{q}_1)\,\vec{q}_1 - (\vec{v}_i^{\,T}\vec{q}_2)\,\vec{q}_2 - \cdots$$

$$= \vec{v}_i - \left(\vec{v}_i^{\,T}\frac{\vec{z}_1}{\|\vec{z}_1\|}\right)\frac{\vec{z}_1}{\|\vec{z}_1\|} - \left(\vec{v}_i^{\,T}\frac{\vec{z}_2}{\|\vec{z}_2\|}\right)\frac{\vec{z}_2}{\|\vec{z}_2\|} - \cdots$$

$$= \vec{v}_i - \frac{\vec{v}_i^{\,T}\vec{z}_1}{\|\vec{z}_1\|^2}\,\vec{z}_1 - \_\_\_ \ \_\_\_ \ \_\_\_$$

$$\left\{\begin{array}{l} \vec{z}_1,\quad \vec{z}_2,\ \vec{z}_3 \ \cdots\cdots \\ \downarrow \\ \vec{q}_1 \quad \vec{q}_2,\ \cdots\cdots \end{array}\right.$$

$\vec{z}_1$

$\downarrow$

$\vec{q}_1$

$\downarrow$

$\vec{z}_2$

$\downarrow$

$\vec{q}_2$

$\vdots$