Walkthrough

Wednesday, April 14, 2021 1:09 PM

EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet **Discussion 12B**

1. Geometric interpretation of the SVD

In this exercise, we explore the geometric interpretation of matrix transformations and how this connects to the SVD. We consider how a real 2×2 matrix acts on the unit circle, transforming it into an ellipse. It turns out that the principal semiaxes of the resulting ellipse are related to the singular values of the matrix, as well as the vectors in the SVD.

diagonal matrix

$$\underline{\underline{A}} = \underline{U}\underline{\Sigma}\underline{V'}$$

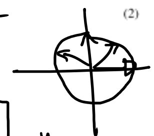
orthonormal matrices

(a) Consider the real 2×2 matrix

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}. \tag{1}$$

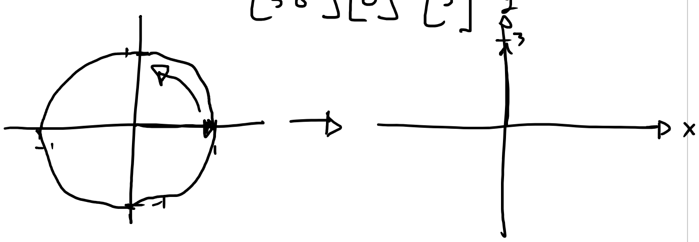
Also consider the unit circle in $\mathbb{R}^2$,

$$S = \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \middle| 0 \le \theta < 2\pi \right\}. \tag{2}$$



Plot the transformed circle, $\underline{AS}$, on the $\underline{\mathbb{R}^2 \text{ plane}}$.

$$\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



$$\underline{U = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}} \quad \underline{\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}} \quad \boxed{V^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$V^T \vec{v}_1 = \begin{bmatrix} - \vec{v}_1^T - \\ - \vec{v}_2^T - \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) Now let's consider how this transformation looks in the lens of the SVD. The SVD for matrix A is:

$$A = U\Sigma V^\top = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\boxed{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}, \tag{3}$$

$$A\vec{x} = U\Sigma V^\top \vec{x} = U\left(\Sigma\left(\underline{V^\top \vec{x}}\right)\right). \tag{4}$$

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad V^T \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let's start by examining the effects of each of these matrices one at a time, right to left, in the same order that they would be applied to a vector $\vec{x}$.

What does the unit circle look like after being transformed by just $V^\top$? Plot $S_1 = V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what does $V^\top$ do to any given $\vec{x}$.

V^T is orthonormal

1) it doesn't change the magnitude of a vector
2) it doesn't change the magnitude of $\measuredangle$ btw vectors

→ reflection
→ rotation
→ both

In this case
V^T: reflected across y axis.

(c) What does the unit circle look like after being transformed by $\Sigma V^\top$? Plot $S_2 = \Sigma V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what is the Σ matrix doing to any given $V^\top \vec{x}$?

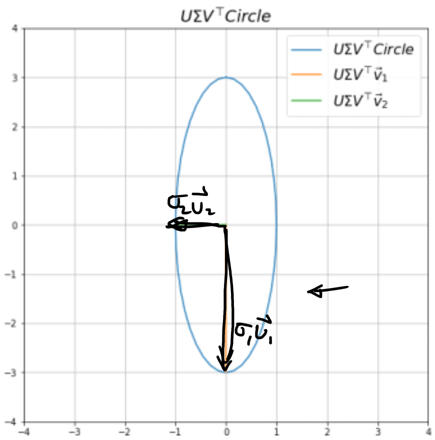
$$\Sigma \underbrace{V^T\vec{x}}_{\vec{s}} \qquad \boxed{\Sigma}\vec{s} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 3s_1 \\ 1s_2 \end{bmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix}$$

(d) What does the unit circle look like after being transformed by $U\Sigma V^\top$? Plot $S_3 = U\Sigma V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what is the U matrix doing to any given $\Sigma V^\top \vec{x}$?

$U(\Sigma V^\top \vec{x})$

orthonormal matrix

• reflect or rotate



(e) Consider the columns of the matrices U, V from the SVI
$\mathbb{R}^2$. Let $U = (\vec{u_1}\ \vec{u_2})$, $V = (\vec{v_1}\ \vec{v_2})$. Let σ_1, σ_2 be the sin
In your plot of AS, draw the vectors $\sigma_1\vec{u_1}$ and $\sigma_2\vec{u_2}$ from
to geometrically?

$\vec{u}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\quad \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\sigma_1 = 3 \quad \sigma_2 = 1$

$U\Sigma \underbrace{V^\top \vec{v}_1}$

$U\Sigma \ \vec{e}_1$

$U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$U \sigma_1 \vec{e}_1$

$\begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} = \sigma_1 \vec{u}_1$

$A\vec{v}_1 = \sigma_1 \vec{u}_1$

$A\vec{v}_2 = \sigma_2 \vec{u}_2$

(f) Repeat parts (b-e) for the following matrices, and note down any interesting things you notice.

$60°$

i. A rotation matrix, $A_1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

ii. A diagonal matrix, $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

$\begin{array}{ccc} U & \Sigma & V^\top \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$

iii. A symmetric matrix, $A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

iv. A matrix with non-trivial nullspace, $A_4 = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$.

$\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

v. An arbitrary matrix, $A_5 = \begin{bmatrix} 1.6 & 2.4 \\ -0.4 & -1 \end{bmatrix}$.

Contributors:

- Lynn Chua.
- Shane Barratt.
- Kuan-Yun Lee.
- Anant Sahai.
- Kareem Ahmad.