Walkthrough

Tuesday, March 2, 2021 2:34 AM

EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet **Discussion 7A**

The first question in this discussion clarifies some results cited in Note 7A and extends the results from Thursday's lecture on Discretization. The second question discusses system evolution over time, and sets the stage for discussing system stability.

1. **A System Governed by Diffential Equations: Piecewise Constant Inputs**

Working through this question will help you better understand differential equations with inputs, and the sampling of a continuous-time system of differential equations into a discrete-time view. These concepts are important for control, since it is often easier to think about doing what we want in discrete-time. This question should initially feel similar to dis03A, and in later subparts, we extend our analysis to the case of a vector differential equation.

(a) Consider the scalar system below:

$$\boxed{\frac{d}{dt}x(t) = \lambda x(t) + u(t).} \quad (1)$$

Further suppose that our input $u(t)$ of interest is piecewise constant over durations of width Δ. This is the same case we considered in dis03A. In other words:

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in [i\Delta, (i+1)\Delta). \quad (2)$$

Similarly, for $x(t)$.

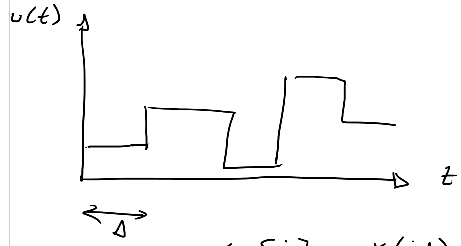
$$x_d[i] = x(i\Delta).$$

Given that we start at $x(i\Delta)$, where do we end up at $x((i+1)\Delta)$?

$$\frac{d}{dt}\left(\alpha e^{\lambda(t-i\Delta)} + \beta\right) = \lambda\left(\alpha e^{\lambda(t-i\Delta)} + \beta\right) + u_d[i]$$

$$\cancel{\lambda\alpha e^{\lambda t}} = \cancel{\lambda\alpha e^{\lambda t}} + \lambda\beta + u_d[i]$$

$$\beta = -u_d[i]/\lambda$$



$$x_d[i] = x(i\Delta)$$

$$x(i\Delta) = x_d[i] \longrightarrow x((i+1)\Delta) = x_d[i+1]$$

from $i\Delta$ to $(i+1)\Delta$

$$\frac{d}{dt}x(t) = \lambda x(t) + \underbrace{u_d[i]}_{\text{constant}}$$

$$x(t) = \alpha e^{\lambda(t-i\Delta)} + \beta$$

$$x(t) = \alpha e^{\lambda(t-i\Delta)} + \left(\frac{-u_d[i]}{\lambda}\right)$$

$$x(i\Delta) = \alpha e^{\cancel{\lambda(i\Delta - i\Delta)}} - \frac{u_d[i]}{\lambda}$$

$$\alpha = x(i\Delta) + u_d[i]/\lambda = x_d[i] + \frac{1}{\lambda}u_d[i]$$

$$x(t) = \left(x_d[i] + \frac{1}{\lambda}u_d[i]\right)e^{\lambda(t-i\Delta)} - \frac{1}{\lambda}u_d[i]$$

$$x((i+1)\Delta) = x_d[i]e^{\lambda\Delta} + \frac{1}{\lambda}u_d[i]e^{\lambda\Delta} - \frac{1}{\lambda}u_d[i]$$

$$\boxed{x_d[i+1] = x_d[i]\underbrace{e^{\lambda\Delta}}_{a} + \underbrace{\frac{e^{\lambda\Delta}-1}{\lambda}}_{b} u_d[i]}$$

(b) Now that we've found a one-step recurrence for $x_d[i+1]$ in terms of $x_d[i]$, we want to get an expression for $x_d[i]$ in terms of the original value $x(0) = x_d[0]$, and all the inputs u. This is so that we can eventually convert this function for $x_d[i]$ into a function for $x(t)$.

Unroll the implicit recursion you derived in the previous part to write $x_d[i+1]$ as a sum that involves $x_d[0]$ and the $u_d[j]$ for $j = 0, 1, \ldots, i$. The idea is to express the value of the discrete system at any arbitrary time solely in terms of where it started, and the accumulating set of inputs until that time.

For this part, to lighten the notation, feel free to just consider the discrete-time system in a simpler form

$$x_d[i+1] = ax_d[i] + bu_d[i] \quad (3)$$

and you don't need to worry about what a and b actually are in terms of λ and Δ.

$$x_d[1] = ax_d[0] + bu_d[0]$$

$$x_d[2] = a(ax_d[0] + bu_d[0]) + bu_d[1]$$

$$x_d[3] = a(a(ax_d[0] + bu_d[0]) + bu_d[1]) + bu_d[2]$$

$$= a^3x_d[0] + a^2bu_d[0] + a^1bu_d[1] + a^0bu_d[2]$$

$$x_d[i] = a^ix_d[0] + \sum_{j=0}^{i-1} a^jbu_d[i-1-j]$$

$$\boxed{x_d[i] = a^ix_d[0] + \left(\sum_{j=0}^{i-1} a^{i-j-1}u_d[j]\right)b}$$

$$x_d[i+1] = a\left(a^ix_d[0] + \left(\sum_{j=0}^{i-1} a^{i-j-1}u_d[j]\right)b\right) + bu_d[i]$$

$$= a^{i+1}x_d[0] + b\sum_{j=0}^{i-1} a^{(i+1)-j-1}u_d[j] + bu_d[i]$$

$$x_d[i+1] = a^{i+1}x_d[0] + b\sum_{j=0}^{(i+1)-1} a^{(i+1)-j-1}u_d[j]$$

✓

$\Delta = 1$

(c) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

$$\boxed{\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)}$$

where $\vec{x}(t)$ is n-dimensional. We will assume that from here onwards, $\Delta = 1$, so that $i\Delta = i$.

Suppose further that the matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We collect the eigenvectors together and form the matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$. *(Hint: What's the significance of this information?)*

If we apply a piecewise constant control input $u_d[i]$ as in (2), and sample the system $\vec{x}(t)$ at time intervals $t = i$, what are the corresponding A_d and $\vec{b}_d$ in:

$$\vec{x}((i+1)\Delta) = \vec{x}(i+1) = A_d\vec{x}(i) + \vec{b}_d u_d[i] \tag{4}$$

$\vec{x}_d[i+1] = \underline{\quad \vec{x}_d[i] + \quad} \; u_d[i]$?

$$\begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{bmatrix} \left(\underbrace{\begin{bmatrix} e^{\lambda_1 \Delta} & & \\ & \ddots & \\ & & e^{\lambda_n \Delta} \end{bmatrix} - I}_{} \right)$$

$\tilde{\vec{x}}(t) = V^{-1}\vec{x}(t)$

$\underline{x(t) = V\tilde{\vec{x}}(t)}$

A has n distinct $\lambda_1 \ldots \lambda_n$ $\vec{v}_1 \ldots \vec{v}_n$

$V^{-1}\frac{d\vec{x}(t)}{dt} = V^{-1}A\vec{x}(t) + V^{-1}\vec{b}u(t)$ $\longrightarrow$

$\tilde{\vec{x}}(t) = V^{-1}\vec{x}(t)$

$\underline{x(t) = V\tilde{\vec{x}}(t)}$

$\frac{d\tilde{\vec{x}}(t)}{dt} = V^{-1}AV\tilde{\vec{x}}(t) + V^{-1}\vec{b}u(t)$

$= \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda} \tilde{\vec{x}}(t) + V^{-1}\vec{b}u(t)$

$\frac{d\tilde{x}_n(t)}{dt} = \lambda_n \tilde{x}_n(t) + (V^{-1}\vec{b})_n u(t)$ ↓

$\tilde{x}_n[i+1] = \tilde{x}_n[i]e^{\lambda_n\Delta} + \frac{e^{\lambda_n\Delta} - 1}{\lambda_n}(V^{-1}\vec{b})u_d[i]$

$\tilde{\vec{x}}[i+1] = \underbrace{\begin{bmatrix} e^{\lambda_1\Delta} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n\Delta} \end{bmatrix}}_{\Lambda_e} \tilde{\vec{x}}[i] + \underbrace{\begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}}_{\Lambda^{-1}(\Lambda_e - I)} V^{-1}\vec{b}u_d[i]$

$\tilde{\vec{x}}[i+1] = \Lambda_e\tilde{\vec{x}}[i] + \Lambda^{-1}(\Lambda_e - I)V^{-1}\vec{b}u_d[i]$

$\vec{x}[i+1] = V\tilde{\vec{x}}[i+1] = V\Lambda_e\tilde{\vec{x}}[i] + V\Lambda^{-1}(\Lambda_e - I)V^{-1}\vec{b}u_d[i]$

$\vec{x}[i+1] = \underbrace{V\Lambda_e V^{-1}}\underline{\vec{x}[i]} + V\Lambda^{-1}(\Lambda_e - I)V^{-1}\vec{b}\,\underline{u_d[i]}$

$A_d = V\Lambda_e V^{-1}$ $\vec{b}_d = V\Lambda^{-1}(\Lambda_e - I)V^{-1}\vec{b}$

$\boxed{\vec{x}[i+1] = A_d\vec{x}[i] + \vec{b}_d u_d[i]}$

$\vec{x}_d[i]$ $x_d[0]$, $u_d[0 \ldots i-1]$

(d) We can write the relationship between the continuous and discrete systems as $\vec{x}(i) = \vec{x}_d[i]$, specifically since $\Delta = 1$. Suppose that $\vec{x}_d[0] = \vec{x}_0$. In the style of part b), unroll the implicit recursion you derived in the previous part to write $\vec{x}_d[i+1]$ as a sum that involves $\vec{x}_0$ and the $u_d[j]$ for $j = 0, 1, \ldots, i$.

For this part, feel free to just consider the discrete-time system in the simpler form

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{5}$$

and you don't need to worry about what A_d and $\vec{b}_d$ actually are in terms of the original parameters.

$\vec{x}_d[i] = A_d^i\vec{x}_d[0] + \left(\sum_{j=0}^{i-1} A_d^{i-j-1} u_d[i]\right)\vec{b}_d$

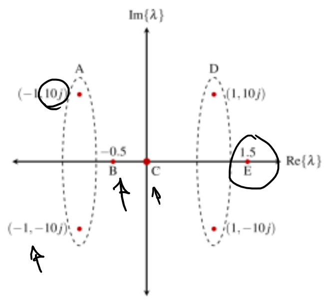
$\vec{x}_d[i+1] = A_d\left(A_d^i\vec{x}_d[0] + \left(\sum_{j=0}^{i-1} A_d^{i-j-1} u_d[i]\right)\vec{b}_d\right) + \vec{b}_d u_d[i]$

$= A_d^{i+1}\vec{x}_d[0] + \left(\sum_{j=0}^{i-1} A^{i+1-j-1} u_d[i]\right)\vec{b}_d + \vec{b}_d u_d[i]$

$\Delta\vec{x}_d[i+1] = A_d^{(i+1)}\vec{x}_d[0] + \left(\sum_{j=0}^{(i+1)-1} A^{(i+1)-j-1} u_d[i]\right)\vec{b}_d$ ✓

2. Continuous-time System Responses

We have a differential equation $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$, where A is diagonal and has eigenvalues λ. For systems A, D, which have more than 1 eigenvalue, this equation is a vector differential equation; in the other cases (B, C, E), it is a scalar differential equation. For each set of λ values plotted on the real-imaginary axis, sketch $\vec{x}_1(t)$ with an initial condition of $\vec{x}_1(0) = 1$. In the scalar case, $\vec{x}_1(t) \equiv x_1(t) \equiv x(t)$.



$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$$

$$\frac{dx_1(t)}{dt} = \lambda_1 x_1(t)$$

$$x_1(t) = x_1(0)e^{\lambda t}$$

$$= x_1(0)e^{\lambda_r t}e^{j\lambda_i t}$$

$$= x_1(0)e^{\lambda_r t}\left(\cos(\lambda_i t) + j\sin(\lambda_i t)\right)$$

real part, imaginary part

$$\lambda = \lambda_r + j\lambda_i$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$Re\{x_1(t)\} = x_1(0)e^{\lambda_r t}\cos(\lambda_i t)$$

$$Im\{x_1(t)\} = x_1(0)e^{\lambda_r t}\sin(\lambda_i t)$$

$\lambda_r > 0$: grow unstable

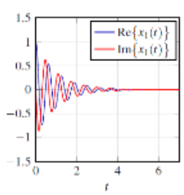
$\lambda_r < 0$: decay stable

$\lambda_r = 1$

$\lambda_i = 0 \rightarrow$ no oscillations

$\lambda_i \neq 0 \rightarrow$ oscillations

A



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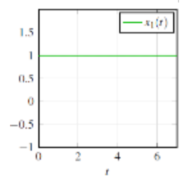

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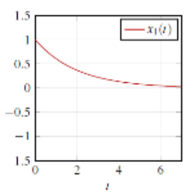
B



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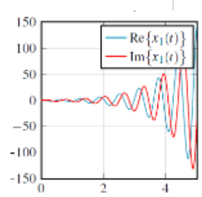

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