
EECS 16B Designing Information Devices and Systems II
 Spring 2021 UC Berkeley

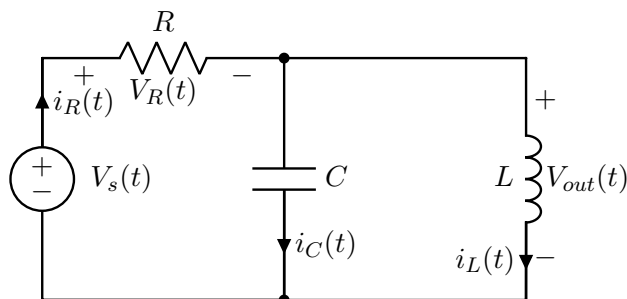
Homework 15

This homework is optional and we are releasing it with solutions. These problems are selected since they might be useful for reviewing for the final.

1. Phasor-Domain Circuit Analysis

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{out}(t)$.



The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2} \cos\left(100t - \frac{\pi}{4}\right)$$

$$R = 5 \Omega$$

$$L = 50 \text{ mH}$$

$$C = 2 \text{ mF}$$

- Give the amplitude V_0 , oscillation frequency ω , and phase ϕ of the input voltage V_s .
- Transform the circuit into the phasor domain. **What are the impedances of the resistor, capacitor, and inductor? What is the phasor \tilde{V}_S of the input voltage $V_s(t)$?**
- Use the circuit equations to **solve for \tilde{V}_{out}** , the phasor representing the output voltage.
- Convert the phasor \tilde{V}_{out} back to get the time-domain signal $V_{out}(t)$.**

2. Latch

The circuit below is a type of latch, which is one of the fundamental components of memory in many digital systems. The latch is a bistable circuit, which means that there are two possible stable states: one representing a stored '1' bit and the other a stored '0' bit.

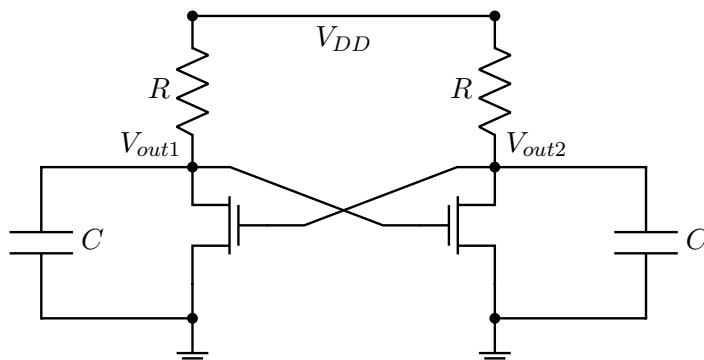


Figure 1: Simplified latch: the gate capacitances have been pulled out explicitly.

- (a) To get a basic understanding of the stable operating points for the latch, consider the following simplified circuit using the pure switch model for MOSFETs (and a threshold voltage of $\frac{V_{DD}}{2}$).

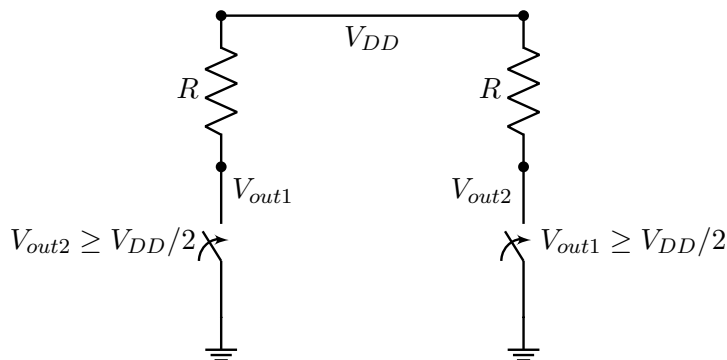


Figure 2: Pure switch model for a latch

First assume that $V_{out1} = 0$. What is V_{out2} ? Are the left and right switches open or closed?

	Open or V_{DD}	Closed or 0
Left Switch	<input type="radio"/>	<input type="radio"/>
Right Switch	<input type="radio"/>	<input type="radio"/>
V_{out2}	<input type="radio"/>	<input type="radio"/>

Suppose that $V_{out1} = V_{DD}$. What is V_{out2} ? Are the left and right switches open or closed?

	Open or V_{DD}	Closed or 0
Left Switch	<input type="radio"/>	<input type="radio"/>
Right Switch	<input type="radio"/>	<input type="radio"/>
V_{out2}	<input type="radio"/>	<input type="radio"/>

- (b) To get an understanding of latch dynamics, we will now break it down into smaller pieces. Below is one half of the latch circuit.

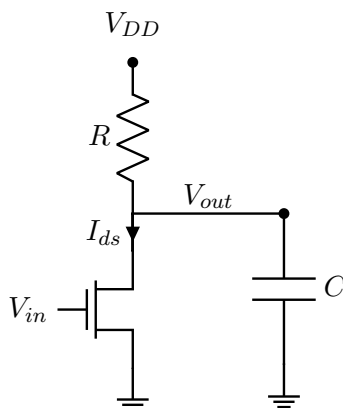
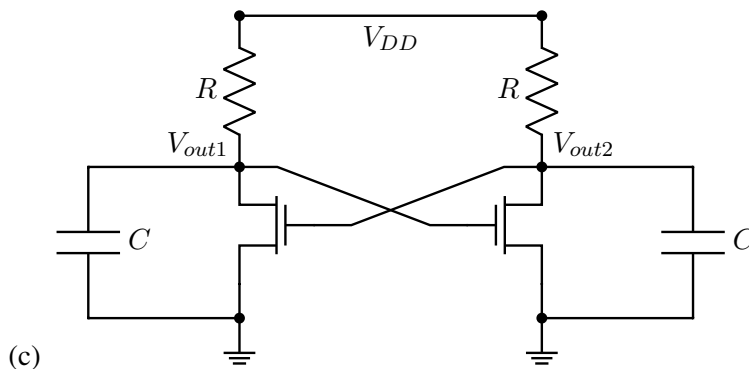


Figure 3: Latch half-circuit

Write a differential equation for the voltage V_{out} in terms of the drain to source current I_{ds} . Treat I_{ds} as some specified input signal and treat the transistor as a current source connected to ground. (i.e. There is no dependence on V_{in} in this part. In this part, treat the I_{ds} as a constant that you are given.)



For this circuit, we care about more detailed analog characteristics of the MOSFETs, so we will model their behavior more accurately as current sources that are controlled by their gate voltages V_{in} with the following equation:

$$I_{ds} = g(V_{in})$$

Where $g(V_{in})$ is a some nonlinear function. **Using this I_{ds} expression together with the result from the previous part, write down a system of differential equations for V_{out1} and V_{out2} in vector form:**

$$\frac{d}{dt} \begin{bmatrix} V_{out1}(t) \\ V_{out2}(t) \end{bmatrix} = \vec{f} \left(\begin{bmatrix} V_{out1}(t) \\ V_{out2}(t) \end{bmatrix} \right).$$

(Hint: Notice that the latch above can be constructed by taking two of the circuit in Figure 3, and connecting the V_{out} of one to the gate V_{in} of the other and vice-versa.)

- (d) For the rest of this problem, assume that your analysis yields the following system of nonlinear differential equations:

$$\begin{bmatrix} \frac{dV_{out1}}{dt} \\ \frac{dV_{out2}}{dt} \end{bmatrix} = \begin{bmatrix} 1 - V_{out1} - g(V_{out2}) \\ 1 - V_{out2} - g(V_{out1}) \end{bmatrix}$$

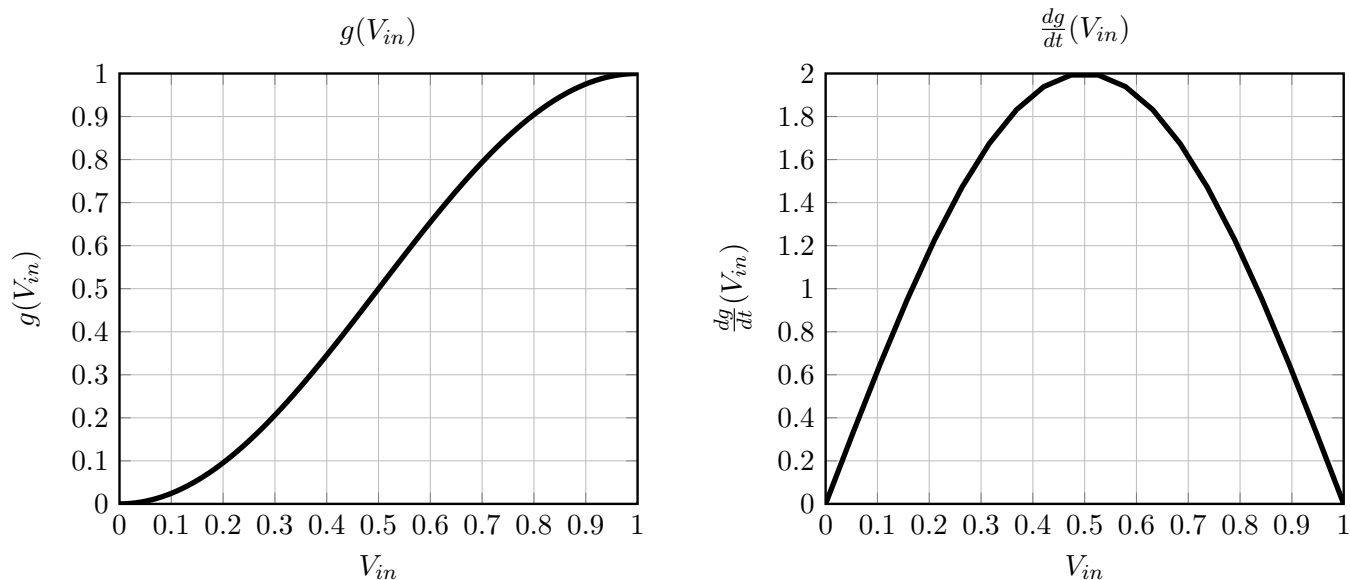
Suppose that you put this latch into an ideal circuit simulator, and measure $g(V_{in})$ and $\frac{dg}{dt}(V_{in})$. The results from these measurements are shown in the graphs below. From your simulations, you also can see that for the following initial conditions, the latch voltages do not change over time:

$$\begin{bmatrix} V_{out1}^* \\ V_{out2}^* \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Use the graphs below to linearize the differential equations around the three operating points.

Write a linearized system of differential equations around each of those operating points $\begin{bmatrix} V_{out1}^* \\ V_{out2}^* \end{bmatrix}$.

For which of the provided $\begin{bmatrix} V_{out1}^* \\ V_{out2}^* \end{bmatrix}$ points is the latch locally stable? For which of the provided points is the latch locally unstable? Why?



3. DFT

Consider the DFT matrix B as defined below

$$B = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{4}} & e^{j2\frac{2\pi}{4}} & e^{j3\frac{2\pi}{4}} \\ 1 & e^{j2\frac{2\pi}{4}} & e^{j4\frac{2\pi}{4}} & e^{j6\frac{2\pi}{4}} \\ 1 & e^{j3\frac{2\pi}{4}} & e^{j6\frac{2\pi}{4}} & e^{j9\frac{2\pi}{4}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}.$$

- (a) The DFT coefficients \vec{F} are related to a vector of samples \vec{f} by the relationship $\vec{f} = B\vec{F}$. In other words, \vec{F} represents \vec{f} in the basis given by the columns of B . Similarly for the DFT coefficients \vec{G} and a vector of samples \vec{g} — they too satisfy the relationship $\vec{g} = B\vec{G}$.

What are the DFT coefficients \vec{H} for $\vec{h} = \alpha\vec{f} + \beta\vec{g}$ in terms of \vec{F} and \vec{G} . Here, α and β are constant real numbers.

- (b) **Explicitly find the DFT coefficients \vec{F} of the vector $\vec{f} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$** i.e. we want \vec{F} so that $\vec{f} = B\vec{F}$.

- (c) **Show that if \vec{f} is a real vector, then:**

- $F[0]$ is always real and so is $F[2]$.
- $F[1] = \overline{F[3]}$.

(HINT: What do you know about B^{-1} ?)

4. DFT and Circuit Filters

You have been introduced to low-pass and high-pass filter circuits that pass some range of input signal frequencies while attenuating other ranges of signal frequency. You have also seen how we can break signals down and view the frequency components of sampled signals using the DFT. In this problem, we will see how we can combine these two bases of knowledge. Throughout this problem, if we have an N -dimensional vector \vec{x} , its DFT coefficients are given by the vector $\vec{X} = F_N \vec{x}$ where the DFT transformation matrix is

$$F_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}3} & \cdots & e^{-j\frac{2\pi}{N}(N-1)} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & e^{-j\frac{2\pi}{N}6} & \cdots & e^{-j\frac{2\pi}{N}(N-1)2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}3(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}$$

and the inverse is $F_N^{-1} = F_N^*$.

- (a) If you sample every Δ seconds and you take N samples, the 0^{th} DFT coefficient $\vec{X}[0]$ corresponds to the DC (or constant) term. The 1^{st} DFT coefficient $\vec{X}[1]$ corresponds to the fundamental frequency $f_0 = \frac{1}{N\Delta}$.

Say you have a signal $v_{in}(t) = \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{9}t\right)$. You take $N = 9$ samples of the function every $\Delta = 1$ second; i.e. at $t = \{0, 1, 2, \dots, 8\}$, forming a 9 element vector of samples \vec{v}_{in} . **What are the DFT coefficients \vec{V}_{in} of the sampled signal \vec{v}_{in} ?**

- (b) You are given the circuit below.

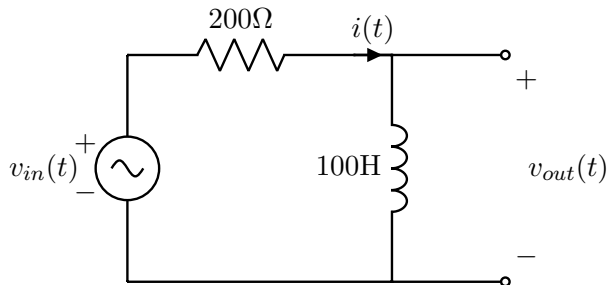
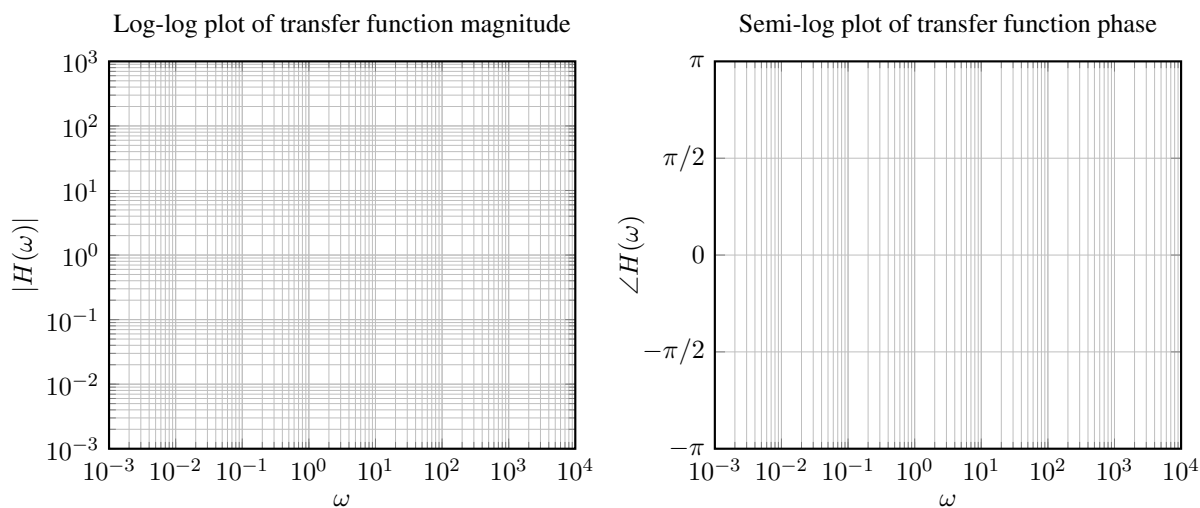
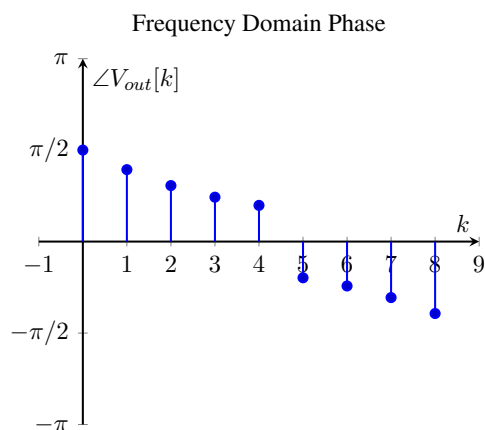
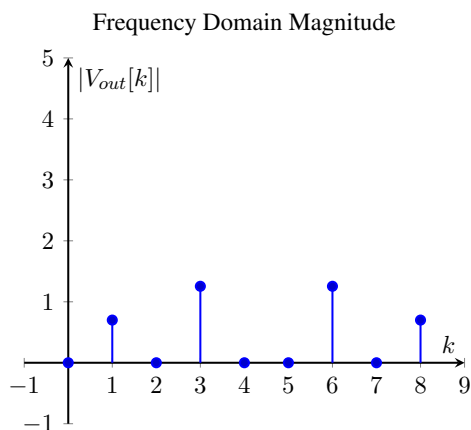
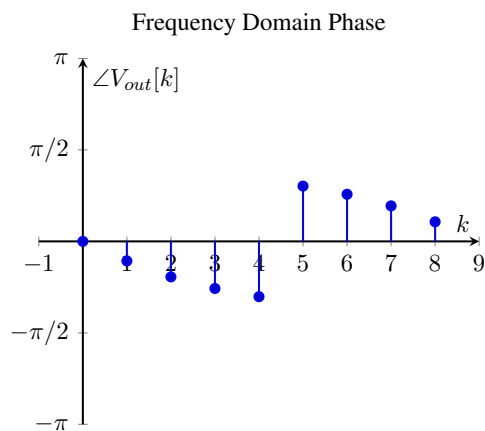
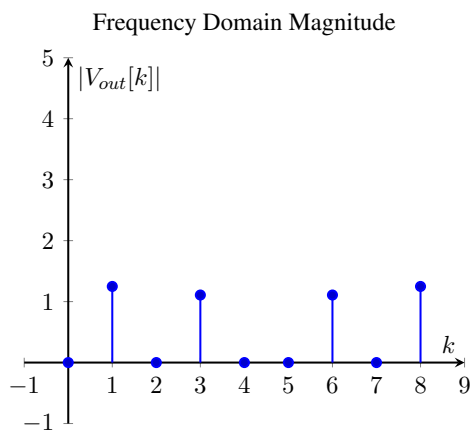


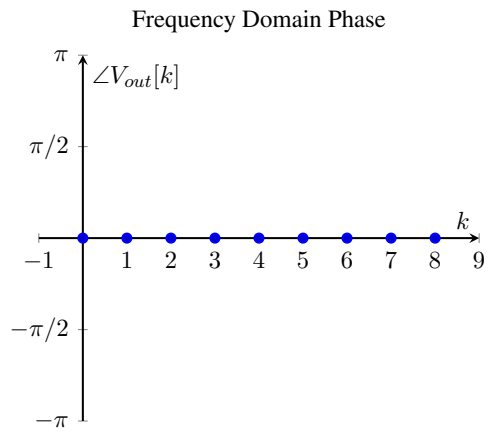
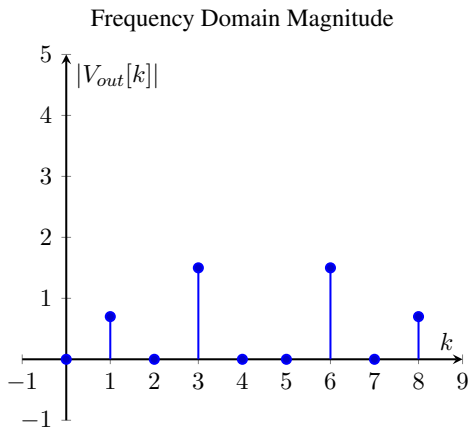
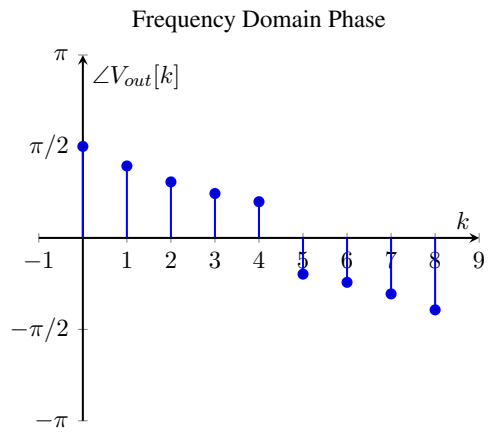
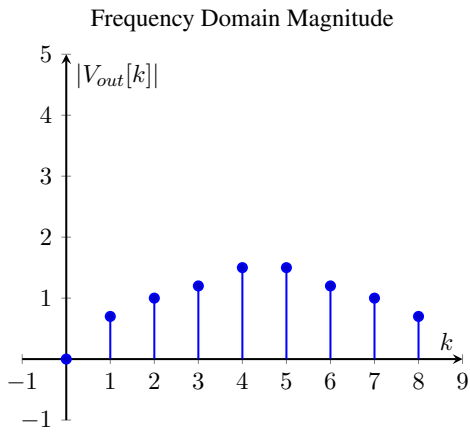
Figure 4: Filter circuit

Is this a high-pass or low-pass filter? What is its cutoff angular frequency, ω_c ? Sketch the piecewise-linear approximations of the magnitude and phase Bode plots of the transfer function $H(\omega) = \frac{\tilde{V}_{out}(\omega)}{\tilde{V}_{in}(\omega)}$ below.



(c) The signal $v_{in}(t) = \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{9}t\right)$ is input into the circuit in Figure 4, giving output signal $v_{out}(t)$. You take $N = 9$ samples of the function $v_{out}(t)$ every $\Delta = 1$ seconds; i.e. at $t = \{0, 1, 2, \dots, 8\}$, forming a 9 element vector of samples \vec{v}_{out} . We have given you several possible plots below that may represent the DFT coefficients \vec{V}_{out} of the sampled signal \vec{v}_{out} . **For each of the four candidate solutions, circle the statement which is true. Provide a one-sentence explanation for your choice in the box provided.** *Reminder: $\omega = 2\pi f$.*
(HINT: Exactly one of the candidate solutions below is correct. Consequently, no precise numerical calculations are required to get full credit.)





5. Adapting Proofs to the Complex Case

At many points in the course, we have made assumptions that various matrices or eigenvalues are real while discussing various theorems. If you have noticed, this has always happened in contexts where we have invoked orthonormality during the proof or statement of the result. Now that you understand the idea of orthonormality for complex vectors, and how to adapt Gram-Schmitt to complex vectors, you can go back and remove those restrictions. This problem asks you to do exactly that.

Unlike many of the problems that we have given you in 16A and 16B, this problem has far less hand-holding — there aren't multiple parts breaking down each step for you. Fortunately, you have the existing proofs in your notes to work based on. So this problem has a secondary function to help you solidify your understanding of these earlier concepts ahead of the final exam.

There is one concept that you will need beyond the idea of what orthogonality means for complex vectors as well as the idea of conjugate-transposes of vectors and matrices. The analogy of a real symmetric matrix S that satisfies $S = S^T$ is what is called a Hermitian matrix H that satisfies $H = H^*$ where $H^* = \overline{H^T}$ is the conjugate-transpose of H .

- (a) The upper-triangularization theorem for all (potentially complex) square matrices A says that there exists an orthonormal (possibly complex) basis V so that V^*AV is upper-triangular.

Adapt the proof from the real case with assumed real eigenvalues to prove this theorem.

Feel free to assume that any square matrix has an (potentially complex) eigenvalue/eigenvector pair. You don't need to prove this. But you can make no other assumptions.

(HINT: Use the exact same argument as before, just use conjugate-transposes instead of transposes.)

Congratulations, once you have completed this part you essentially can solve all systems of linear differential equations based on what you know, and you can also complete the proof that having all the eigenvalues being stable implies BIBO stability.

- (b) The spectral theorem for Hermitian matrices says that a Hermitian matrix has all real eigenvalues and an orthonormal set of eigenvectors.

Adapt the proof from the real symmetric case to prove this theorem.

(HINT: As before, you should just leverage upper-triangularization and use the fact that $(ABC)^ = C^*B^*A^*$. There is a reason that this part comes after the first part.)*

- (c) The SVD for complex matrices says that any rectangular (potentially complex) matrix A can be written as $A = \sum_i \vec{u}_i \sigma_i \vec{w}_i^*$ where σ_i are real positive numbers and the collection $\{\vec{u}_i\}$ are orthonormal (but potentially complex) as well as $\{\vec{w}_i\}$.

Adapt the derivation of the SVD from the real case to prove this theorem.

Feel free to assume that A is wide. (Since you can just conjugate-transpose everything to get a tall matrix to become wide.)

*(HINT: Analogously to before, you're going to have to show that the matrix A^*A is Hermitian and that it has non-negative eigenvalues. Use the previous part. There is a reason that this part comes after the previous parts.)*

6. Rank 1 Decomposition

In this problem, we will decompose a few images into linear combinations of rank 1 matrices. Remember that outer product of two vectors $\vec{s}\vec{g}^T$ gives a rank 1 matrix. It has rank 1 because clearly, the column span is one-dimensional — multiples of \vec{s} only — and the row span is also one dimensional — multiples of \vec{g}^T only.

- (a) Consider a standard 8×8 chessboard shown in Figure 5. Assume that black colors represent -1 and that white colors represent 1 .

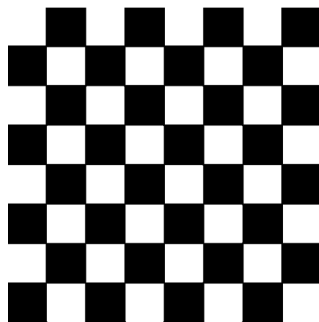


Figure 5: 8×8 chessboard.

Hence, that the chessboard is given by the following 8×8 matrix C_1 :

$$C_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Express C_1 as a linear combination of outer products. *Hint: In order to determine how many rank 1 matrices you need to combine to represent the matrix, find the rank of the matrix you are trying to represent.*

- (b) For the same chessboard shown in Figure 5, now assume that black colors represent 0 and that white colors represent 1 .

Hence, the chessboard is given by the following 8×8 matrix C_2 :

$$C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Express C_2 as a linear combination of outer products.

(c) Now consider the Swiss flag shown in Figure 6. Assume that red colors represent 0 and that white colors represent 1.

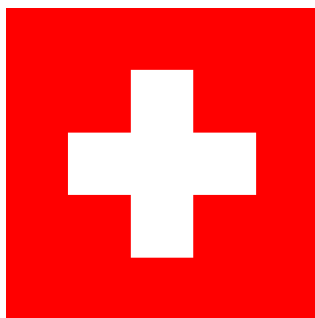


Figure 6: Swiss flag.

Assume that the Swiss flag is given by the following 5×5 matrix S :

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Furthermore, we know that the Swiss flag can be viewed as a superposition of the following pairs of images:



Figure 7: Pairs of images - Option 1



Figure 8: Pairs of images - Option 2

Express the S in two different ways: i) as a linear combination of the outer products inspired by the Option 1 images and ii) as a linear combination of outer products inspired by the Option 2 images.

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