EECS 16B Module 3 Lecture 1(?) ish

Today: Apr 1, 2021. 1-1 meetings in OH.

- Minimum energy control
- Spectral theorem for symmetric matrices Recap.
- Singular Value Decomposition. (SVD).

Min energy control.

- Stability
- Controllability
- Efficiency

Want $\vec{x}[100] = \vec{d}$ (fixed). $\rightsquigarrow d \in \mathbb{R}^{10}$

$\vec{x} \in \mathbb{R}^{10}$

$$\boxed{\begin{aligned} \vec{x}[k+1] &= \underbrace{A}_{10\times 10}\vec{x}[k] + \underbrace{\vec{b}}_{10\times 1}u[k]. \\ \vec{x}[0] &= 0. \end{aligned}}$$

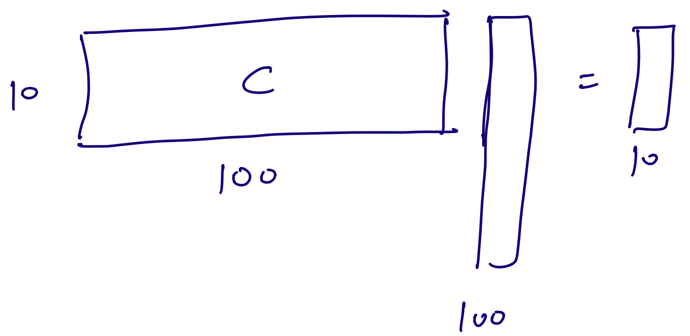
$$\vec{x}[100] = A^{99}\vec{b}\,u[0] + A^{98}\vec{b}\,u[1] + \cdots + A\vec{b}\,u[98] + \vec{b}\,u[99]$$

$$= {}_{10}\underbrace{\begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{99}\vec{b} \end{bmatrix}}_{100} \begin{bmatrix} u[99] \\ u[98] \\ \vdots \\ u[0] \end{bmatrix}$$

$$= \boxed{\underbrace{C}_{10\times 100} \cdot \underbrace{\vec{u}}_{100\times 1} = \vec{d}}$$

<u>Goal</u> : find $\vec{u}$ that minimizes

$$\underset{C \cdot \vec{u} = \vec{d}}{\arg\min} \; \|\vec{u}\|^2$$

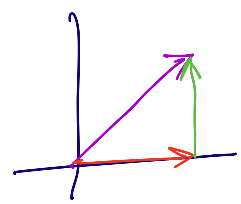


infinite.

Say $\vec{u}_{sol}$ is a solution.
→ infinite optims

Which means.

$$C \cdot \vec{u}_{sol} = \vec{d}$$



<u>Say</u> <u>Hypothetically</u> we knew that

$$\vec{u}_{sol} = \vec{u}_{null} + \vec{u}_{other}$$

$\vec{u}_{null}$, $\vec{u}_{other}$

① . $C \cdot \vec{u}_{null} = \vec{0}$ Nullspace of C

② $\vec{u}_{null} \perp \vec{u}_{other}$, $\langle \vec{u}_{null}, \vec{u}_{other} \rangle = 0$

→ orthogonal to nullspace

$$\vec{d} = C\left[\vec{u}_{sol}\right] = C\left[\vec{u}_{null} + \vec{u}_{other}\right]$$

$$= C \cdot \vec{u}_{null} + C \cdot \vec{u}_{other}$$

$$= 0 + C \cdot \vec{u}_{other}$$

$\Rightarrow$ If $\vec{u}_{sol}$ is a solution, then, $\vec{u}_{other}$ is also a solution.

$$\|\vec{u}_{sol}\|^2 = \langle \vec{u}_{sol}, \vec{u}_{sol} \rangle$$

$$= \langle \vec{u}_{null} + \vec{u}_{other}, \vec{u}_{null} + \vec{u}_{other} \rangle$$

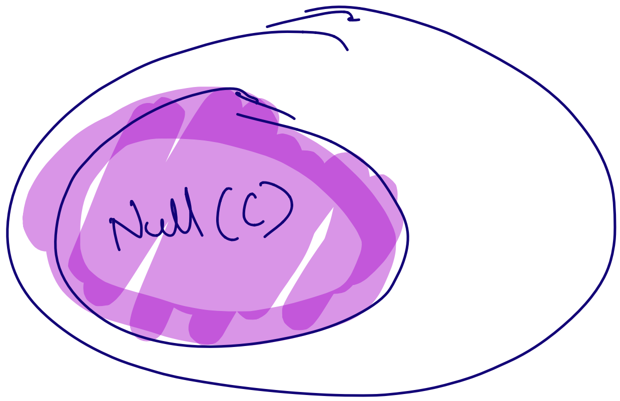
$$= \langle \vec{u}_{null}, \vec{u}_{null} \rangle + \langle \vec{u}_{other}, \vec{u}_{other} \rangle$$

$$+ \cancelto{0}{\langle \vec{u}_{null}, \vec{u}_{other} \rangle}$$

$$+ \cancelto{0}{\langle \vec{u}_{other}, \vec{u}_{null} \rangle}$$

$$\|\vec{u}_{sol}\|^2 = \|\vec{u}_{null}\|^2 + \|\vec{u}_{other}\|^2$$

$$\boxed{\begin{array}{l}\text{We always have a preference for } \vec{u}_{other}. \\ \|\vec{u}_{sol}\|^2 \geq \|\vec{u}_{other}\|^2\end{array}}$$



We want no component of v in the nullspace of C.

Recap: Symmetric matrices + Spectral thm.

$S = S^T$ $\qquad S \in \mathbb{R}^{n \times n}$

Then: $\quad S = V \Lambda V^{-1}$

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$ $\vec{v}_i$ are eigenvecs

$$\|\vec{v}_i\|^2 = 1$$

$$\langle \vec{v}_i, \vec{v}_j \rangle = 0 \quad \text{for } i \neq j$$

Λ is diagonal and has real eigenvalues on the diagonal.

$\lambda_1, \lambda_2, \cdots \lambda_n$.

Arrange these such that

$$\lambda_1, \lambda_2 \cdots, \lambda_r \neq 0, \quad \lambda_{r+1}, \lambda_{r+2}, \cdots \lambda_n = 0$$

$$S = V \Lambda V^T$$

V orthonormal

$$V^{-1} = V^T$$

$$\Rightarrow SV = V\Lambda$$

$$\Rightarrow S \begin{bmatrix} \underbrace{\vec{u}_1 \cdots \vec{u}_r}_{\text{non-zero evals}} & \underbrace{\vec{u}_{r+1} \cdots \vec{u}_n}_{\text{zero-evals}} \end{bmatrix} = \left[\begin{array}{ccc|cc} \vec{u}_1 & \vec{u}_2 \cdots & \vec{u}_r & \vec{u}_{r+1} \cdots & \vec{u}_n \end{array}\right] \left[\begin{array}{cccc|cccc} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & \lambda_r & & & & \\ & 0 & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{array}\right]$$

Zero evals.

Now: $S \cdot \vec{u}_{r+1} = 0 \cdot \vec{u}_{r+1} = \vec{0}$

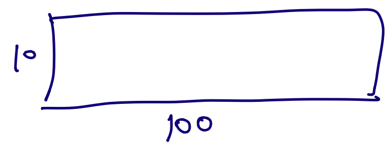
$S \cdot \vec{u}_{r+2} = \vec{0}$

$\vdots$

$S \cdot \vec{u}_n = \vec{0}$

$\Rightarrow \vec{u}_{r+1}, \vec{u}_{r+2}, \ldots \vec{u}_n$ form a basis for the nullspace of S.

SVD.

C : 

C : Rank 10.

$\Longrightarrow$ 10 non-zero evals.
90 evals $= 0$

Find: $C \cdot V = \cancel{V \Lambda}$

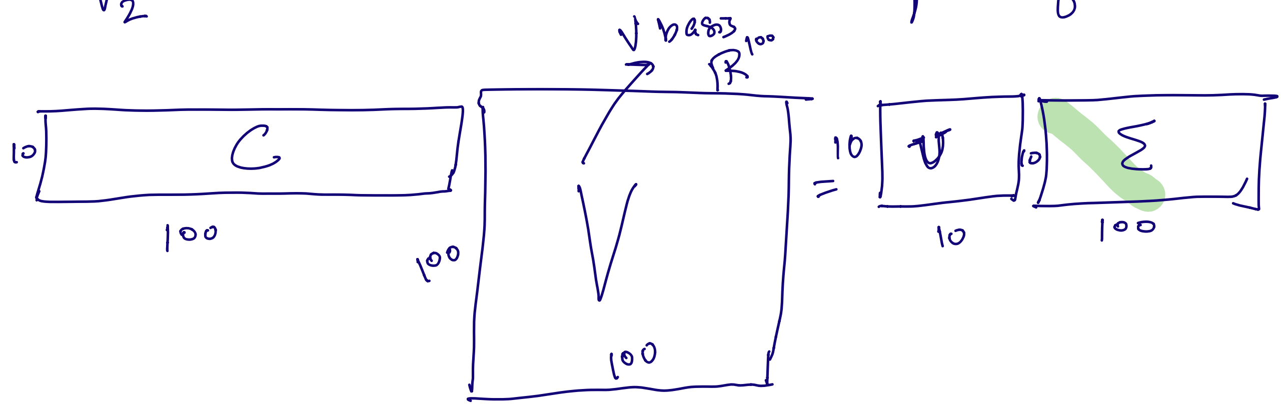
$$C \cdot \underbrace{V}_{\text{orthonormal}} = \underbrace{U}_{\text{orthonormal}} \cdot \Sigma$$

Σ ← almost diagonal

$$C \Big[\underbrace{\overbrace{\vec{v}_1 \ldots \vec{v}_{10}}^{V_1}}_{10} \; \underbrace{\overbrace{\vec{v}_{11} \cdots \vec{v}_{100}}^{V_2}}_{90} \Big]$$

$$= C \left[V_1 \quad V_2 \right]$$

V_2 : forms a basis for the nullspace of C.



Consider: $C^TC = S$

S is symm if $S^T = S$.

Is C^TC symmetric?

$$(C^TC)^T = C^T(C^T)^T = C^T \cdot C.$$

$\Rightarrow$ C^TC is always a symmetric matrix

Special property ①: C^TC:

$$\text{Null}(C^TC) = \text{Null}(C)$$

$\rightarrow$ Refer to 16A.

(1) If $C^TC \cdot \vec{v} = 0 \Rightarrow C \cdot \vec{v} = 0$

(2) If $C \cdot \vec{v} = 0 \Rightarrow C^TC \cdot \vec{v} = 0$

Special property ②: Eigenvalues of C^TC: real

Always non negative evals!!

$\boxed{C^TC = S}$

Proof: λ is an e-val of $C^TC = S$

$S \cdot \vec{v} = \lambda \cdot \vec{v}$ $\quad$ $\vec{v}$ is e-vector.

$C^TC \cdot \vec{v} = \lambda \cdot \vec{v}$

Try: multiply both sides by $\vec{v}^T$

$$\vec{v}^T \cdot C^T \cdot C \cdot \vec{v} = \vec{v}^T \cdot \lambda \cdot \vec{v}$$

$$(C\vec{v})^T (C \cdot \vec{v}) = \lambda \, \|\vec{v}\|^2$$

$$\|C\vec{v}\|^2 = \lambda \cdot \|\vec{v}\|^2$$

$$\lambda = \frac{\|C\vec{v}\|^2}{\|\vec{v}\|^2} \geq 0$$

Any e-val of C^TC is always non-negative.

Consider: $S = C^TC \in \mathbb{R}^{100 \times 100}$ $C \in \mathbb{R}^{10 \times 100}$

V is the matrix of e-vectors of S.

$$\underbrace{S}_{100 \times 100} \underbrace{V}_{100 \times 100} = V \underbrace{\Lambda}_{100 \times 100}.$$

$$V = \left[\underbrace{\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_{10}}_{\text{non-zero-evals}} \;\middle|\; \underbrace{\vec{v}_{11} \; \cdots \; \vec{v}_{100}}_{\text{Basis for nullspace of } S} \right]$$

e-vectors for the 0 e-values

- $Null(S) = null(C)$.

$\Rightarrow \quad \vec{u}_{11}, \vec{u}_{12}, \ldots \vec{u}_{100}$ are also a basis for $null(C)$.

Now consider

$$C \cdot V = C \begin{bmatrix} V_{col} & V_{null} \end{bmatrix}$$

V_{null}

V_{col}

basis for the null of C!

$$V = \begin{bmatrix} V_{col} & V_{null} \end{bmatrix}$$

$\boxed{V_{null} \text{ forms a basis for } Null(C)}$

<u>Claim:</u> The columns of $C \cdot V_{col}$ form a basis for the columnspace of C!

$V_{col} = \{\vec{u}_1, \vec{u}_2 \;..\; \vec{u}_{10}\}$ ← e-vecs of non-zero evals

$C \cdot V_{col} = \{C\vec{u}_1, C\vec{u}_2 \;..\; C \cdot \vec{u}_{10}\}$.

① Check linear independance. ✓

Consider: $C\vec{v_i}$, $C\vec{v_j}$

$\langle C\vec{v_i}, C\vec{v_j} \rangle$ ← orthogonal!

$= \vec{v_i}^T C^T C \cdot \vec{v_j}$

$= \vec{v_i}^T \underbrace{S \cdot \vec{v_j}}$

$= \vec{v_i}^T \cdot \lambda_j \vec{v_j} = \lambda_j \vec{v_i}^T \vec{v_j} = 0$

② $Col(C) = \{ \vec{y} \mid \vec{y} = C \cdot \vec{w}$ for some $\vec{w} \}$.

$$\vec{w} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_n \vec{v_n}$$

$$\vec{y} = C \cdot \vec{w} = C \cdot \sum_{i=1}^{n} \alpha_i \cdot \vec{v_i}$$

$$= \sum_{i=1}^{n} \alpha_i \left(C \cdot \vec{v_i} \right)$$

$$\vec{y} = \sum_{i=1}^{10} \alpha_i \left(C \cdot \vec{v_i} \right)$$

$C \cdot \vec{v_i} = 0$ for $\vec{v_i}$ in $Null(C)$

So for $i > 10$, $C \cdot \vec{v_i} = 0$