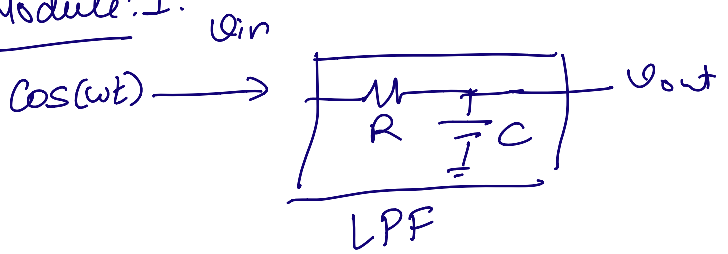
EECS 16B   Module 3, Lecture 8.

Today: DFT: Discrete Fourier Transform.

- New perspective: frequency domain
  - → Signal processing
  - → Fourier features.
  - → Teaser for ideas from 120, 170, 189, 123.
    FFT

Module 1:



$\omega_0 = \frac{1}{RC}$

Sinusoids / Cosines : Sums of complex exponentials.

$$\cos(\omega t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

Some signal: $x[1], x[2] \dots \quad x[N]$ ← real valued samples.

N : finite fixed time.

$$\vec{x} = \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ \vdots \\ x[N] \end{bmatrix} \in \mathbb{R}^N$$

$\vec{x}$ can be represented in a basis of complex exponentials.

e.g. $x[n] = \sin[\omega n]$.

$$\sin[\omega n] = \frac{1}{2j} e^{j\omega n} - \frac{1}{2j} e^{-j\omega n}$$

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{2j} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega \cdot 1} \\ \vdots \\ e^{j\omega(N-1)} \end{bmatrix} - \frac{1}{2j} \begin{bmatrix} e^{j\omega 0} \\ e^{-j\omega 1} \\ \vdots \\ e^{-j\omega(N-1)} \end{bmatrix}$$

Before computing DFT, first some complex linear algebra.

Complex vector: $\vec{z} \in \mathbb{C}^N$ is a length N complex vector if every entry is a ~~comp~~ complex number.

e.g. $\begin{bmatrix} 1 \\ j \end{bmatrix} \in \mathbb{C}^2$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{C}^2$

$\begin{bmatrix} 1 \\ 1+j \end{bmatrix} \in \mathbb{C}^2$ ... etc. $\vec{z} = \begin{bmatrix} 1+j \\ 4+2j \end{bmatrix}$

- Norm $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

Consider: $\vec{z} = [j]$

$j \times j = -1$

$$\boxed{\begin{aligned} \|\vec{z}\|^2 &= |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \\ &= \bar{z}_1 \cdot z_1 + \bar{z}_2 \cdot z_2 + \cdots + \bar{z}_n \cdot z_n . \end{aligned}}$$

$\boxed{z_1 = a + bj}$

$\bar{z}_1 = a - bj$

$$\begin{aligned} \bar{z}_1 \cdot z_1 &= (a-bj)(a+bj) \\ &= a^2 - (bj)^2 \\ &= a^2 + b^2 \end{aligned}$$

Complex inner product $\qquad \vec{v} \in \mathbb{C}^N, \ \vec{u} \in \mathbb{C}^N$

$$\langle \vec{v}, \vec{u} \rangle = \left(\bar{\vec{u}}\right)^T \vec{v}$$

conjugate transpose.

$$= \vec{u}^{\,*} \vec{v}$$

e.g. $\vec{v} = \begin{bmatrix} 1 \\ j \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1+j \\ 2 \end{bmatrix}.$

$$\langle \vec{v}, \vec{u} \rangle = \left\langle \begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} 1+j \\ 2 \end{bmatrix} \right\rangle$$

$$= \begin{bmatrix} 1-j & 2 \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$= 1 - j + 2j$

$= 1 + j$ $\quad$ Scalar
Complex.

e.g. $\left\langle \begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix} \right\rangle$

$= \begin{bmatrix} -j & 1 \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix}$

$= -j + j = 0$

Two complex vectors are orthogonal if their inner product is 0.

Note: Order of inner product. $\quad (AB)^T = B^T A^T$

$< \vec{v}, \vec{u} > = \vec{u}^* \vec{v}$

$\left(< \vec{v}, \vec{u} > \right)^* = \left(\vec{u}^* \vec{v} \right)^*$

$= \left(\vec{v}^* \right) \left(\vec{u}^* \right)^*$

$$= \vec{v}^* \vec{u}$$

$$= \langle \vec{u}, \vec{v} \rangle$$

Change in order of inner products $\rightarrow$ conjugate relationship

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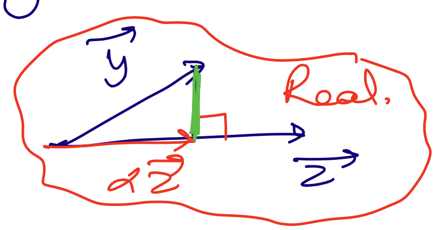
<u>Complex Projections</u> : Project $\vec{y}$ onto $\vec{z}$

$\vec{y}, \vec{z} \in \mathbb{C}^N$



Find α such that

$$(\vec{y} - \underbrace{\alpha\vec{z}}_{\text{projection of } \vec{y} \text{ onto } \vec{z}}) \perp \vec{z}$$

$$\langle \vec{y} - \alpha\vec{z}, \vec{z} \rangle = \vec{z}^* (\vec{y} - \alpha\vec{z})$$

$$0 = \vec{z}^* \vec{y} - \alpha \cdot \|\vec{z}\|^2$$

$$\alpha = \frac{\langle \vec{y}, \vec{z} \rangle}{\langle \vec{z}, \vec{z} \rangle}$$

Projection of $\vec{y}$ onto $\vec{z}$: $\underbrace{\dfrac{\langle \vec{y}, \vec{z} \rangle}{\langle \vec{z}, \vec{z} \rangle} \cdot \vec{z}}_{\text{Same as the real case!}}$

Roots of unity.

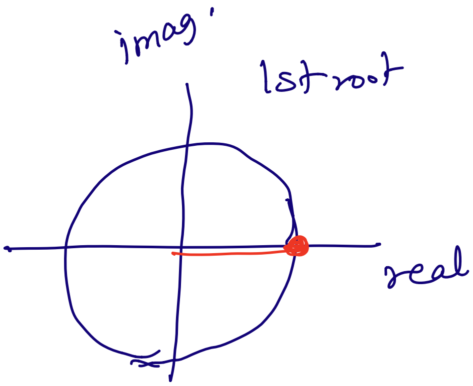
1st root of unity? $\quad z^1 = 1 \quad \Rightarrow z = 1.$

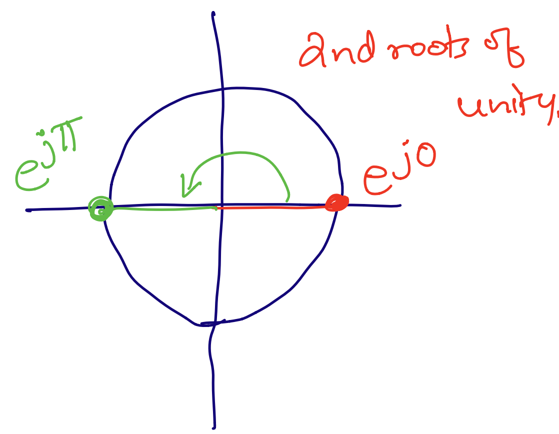
2nd roots of unity? $\quad z^2 = 1 \quad \Longrightarrow z = +1, -1$

$$z = e^{j\frac{2\pi}{2}\cdot 0},\ e^{j\frac{2\pi}{2}\cdot 1}$$

$$e^{j0} = \cos 0 + j\sin 0 = 1$$

$$e^{j\pi} = \cos\pi + j\sin\pi = -1$$



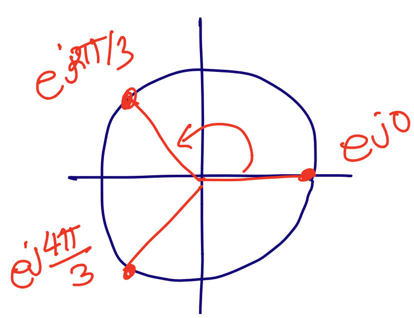


3rd root of unity.

$z^3 = 1 \quad , \quad z = 1$

Find all roots of $\quad z^3 - 1 = 0$

Consider : $\quad e^{j2\pi/3} \quad , \quad e^{j(2\pi/3)\cdot 2}.$



$$\boxed{z = e^{j2\pi/3}}$$

$$z^3 = \left(e^{j2\pi/3}\right)^3 = e^{j2\pi} = \cos 2\pi + j\sin 2\pi$$

$$= 1$$

$$z = e^{j4\pi/3}$$

$$z^3 = e^{j4\pi} = 1.$$

Roots of unity: $z^N - 1 = 0$ N th roots of unity.

$$(z^N - 1) = (z-1)\left(z^{N-1} + z^{N-2} + \cdots + z + 1\right)$$

In general: the roots of this equation are given by:

$$e^{j\frac{2\pi}{N}\cdot k} \quad \text{for } k = \{0, 1, \ldots, N-1\}$$

$$\left(e^{j\frac{2\pi}{N}k}\right)^N = e^{j2\pi k} = 1.$$

Say I have ω, $\omega \in \mathbb{C}$, ω is an Nth root of unity. (1).

ω is a solution to $z^N - 1 = 0$.

$$\Rightarrow \quad w^N - 1 = 0.$$

$$\Rightarrow \underbrace{(w-1)}_{\neq 0} \underbrace{(w^{N-1} + w^{N-2} + \cdots + w + 1)}_{= 0.} = 0.$$

If $w \neq 1 \Rightarrow 1 + w + \cdots + w^{N-1} = 0.$

e.g. $N = 3.$, $w = e^{j 2\pi/3}$.

$$\Rightarrow 1 + e^{j2\pi/3} + e^{j4\pi/3} = 0.$$

" Sums of the roots of unity is 0 ".

Prep work done!

Developing the DFT basis.

Discrete Fourier Transform.

$$U = \frac{1}{\sqrt{N}} \begin{array}{c} \begin{array}{cccccc} \vec{u}_0 & \vec{u}_1 & \cdots & & & \vec{u}_{N-1} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & w^3 & & w^{N-1} \\ \vdots & w^2 & w^4 & w^6 & & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & w^{3(N-1)} & & w^{(N-1)(N-1)} \end{bmatrix} \end{array}$$

$$\underbrace{w = e^{j\frac{2\pi}{N}}}_{N\text{th root of unity.}}$$

Columns of U are the vectors that make up the "DFT" basis.

<u>kth column:</u>

$0 \le k \le N-1.$

$$\begin{bmatrix} 1 \\ \omega^{k} \\ \omega^{2k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix}$$

<u>e.g. $N=3$</u>

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j2\pi/3} & \left(e^{j2\pi/3}\right)^2 \\ 1 & e^{j4\pi/3} & \left(e^{j4\pi/3}\right)^2 \end{bmatrix}$$

$e^{4\pi/3}.$

$$e^{j8\pi/3} = e^{j\left(\frac{6\pi}{3}+\frac{2\pi}{3}\right)} = e^{j\left(2\pi+\frac{2\pi}{3}\right)} = e^{j2\pi}\cdot e^{j\frac{2\pi}{3}} = e^{j2\pi/3}$$

$$= \frac{1}{\sqrt{3}} \begin{array}{c} \begin{array}{ccc} \vec{u}_0 & \vec{u}_1 & \vec{u}_2 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j2\pi/3} & e^{j4\pi/3} \\ 1 & e^{4\pi/3} & e^{j2\pi/3} \end{bmatrix} \end{array}$$

~~2nd~~

(1) All columns are unit norm.

② All columns are orthonormal:

$$\underbrace{\langle \vec{u_0}, \vec{u_1} \rangle} = \begin{bmatrix} 1 & e^{-j\frac{2\pi}{3}} & e^{j\frac{4\pi}{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 1 + e^{-j2\pi/3} + e^{-j4\pi/3}$$

$$= 1 + e^{j4\pi/3} + e^{j2\pi/3}$$

$$= 0$$