Lecture 6

EECS 16B

* Math Recap
* Solving Systems of Diff. Eqns.
  - Diagonalization
* Intro to inductors

EECS 16B Math Recap:

(1) Started with first-order systems:

$$\boxed{\frac{d}{dt} x(t) = \lambda x(t)} \underset{\text{guess \& check uniqueness}}{\Rightarrow} \boxed{x(t) = x(0)\, e^{\lambda t}}$$

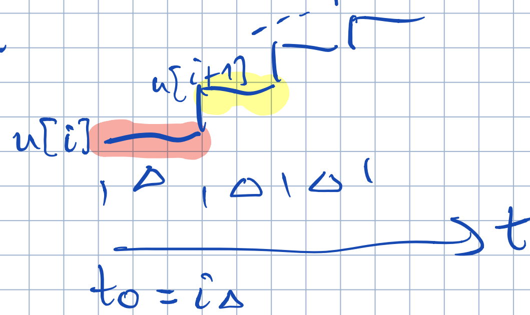
(2) added a constant input

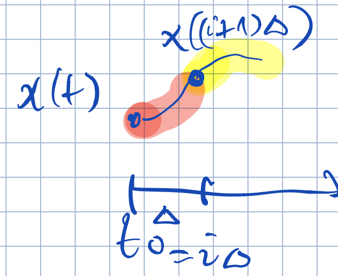
$$\frac{d}{dt} x(t) = \lambda x(t) + u \underset{\text{change of variables}}{\Rightarrow} \boxed{\frac{d}{dt} \tilde{x}(t) = \lambda \tilde{x}(t)}$$

$$\tilde{x} = x + \frac{u}{\lambda} \qquad \boxed{\tilde{x}(t) = \tilde{x}(0) \cdot e^{\lambda t}}$$

$$\boxed{x(t) = x(0) \cdot e^{\lambda t} + \frac{e^{\lambda t} - 1}{\lambda} u}$$

③ extended it to piecewise-constant input:





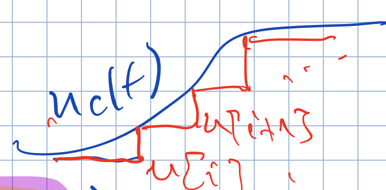
$$\frac{d}{dt} x(t) = \lambda x(t) + u[i] \quad , \quad t \in [i\Delta, (i+1)\Delta)$$

$$\boxed{x(t) = x(i\Delta) \cdot e^{\lambda(t - i\Delta)} + \frac{e^{\lambda(t-i\Delta)} - 1}{\lambda} u(i\Delta)}$$

iterate for i

④ piecewise-wise constant approximates continuous input for $\Delta \to 0$

$$x(t) = x(0) \cdot e^{\lambda t} + \frac{e^{\lambda t} - e^{\lambda \Delta}}{\lambda} \sum_{j=0}^{n-1} e^{-\lambda \Delta j} u_c(j\Delta)$$



$\Delta \to 0$

(Disc. 3A)

$t \geq 0$

as $\Delta \to 0$

$$x(t) = x(0) e^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda \theta} u_c(\theta) d\theta$$

(1) $x(t) = x(0)e^{\lambda t} + e^{\lambda t}\int_0^t e^{-\lambda\theta} u_c(\theta)\, d\theta$

solves: (2) $\frac{d}{dt}x(t) = \lambda x(t) + u_c(t)$, $t \geq 0$

(5) Solved for: $u_c(t) = K \cdot e^{st}$ $\Rightarrow$

$\Rightarrow$ $x(t) = \alpha e^{st}$ $\Rightarrow$

guess & check with (2)

$$\boxed{x(t) = x(0)e^{\lambda t} + \frac{K}{s-\lambda}\left(e^{st} - e^{\lambda t}\right)}$$

use (1) & solve

(6) Solved for: $u_c(t) = K\cos(\omega t)$

(*) guess & check $\rightarrow$ trig identities $\rightarrow$ answer

(**) use (1) & solve

$$x(t) = x(0) \cdot e^{\lambda t} + K\frac{\omega \sin(\omega t) - \lambda\cos(\omega t)}{\lambda^2 + \omega^2} + \frac{K\lambda e^{\lambda t}}{\lambda^2 + \omega^2}$$

RC: $\lambda = -\frac{1}{RC}$

$$V_{out}(t) = \frac{V_{in}}{\sqrt{1+(\omega RC)^2}} \cdot \cos\left(\omega t - \tan^{-1}(\omega RC)\right)$$

7 Systems of differential equations

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \qquad \vec{x}(0) \text{ init condition}$$

Native $\vec{x}$ coordinates:

"Nice" $\vec{\tilde{x}}$ coordinates:

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\
\vec{\tilde{x}} \arrow[u, bend right, "V"'] \arrow[r, "?"'] & \frac{d}{dt}\vec{\tilde{x}} \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$\boxed{V^{-1}AV}$$

$$\vec{x}(t) = V\vec{\tilde{x}}(t)$$

$$\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \frac{d}{dt}V^{-1}\vec{x}(t) = V^{-1}\frac{d}{dt}\vec{x}(t) = V^{-1}A\vec{x} = \underline{V^{-1}AV\vec{\tilde{x}}}$$

want: $V^{-1}AV$ is at least upper-triang.

$$\begin{bmatrix} \ddots & * \\ 0 & \bullet \end{bmatrix} \to 1 \text{ scalar D.E} \to \text{GE}$$

best: $V^{-1}AV$ is diagonal

$$\begin{bmatrix} \bullet & & 0 \\ & \ddots & \\ 0 & & \bullet \end{bmatrix}$$

$\Rightarrow$ all equations are scalar D.E

$\Rightarrow$ know how to solve ☺

Want: $V^{-1}AV = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

How to figure out the right V?

$$V = [\vec{v_1} \; \dots \; \vec{v_n}]$$

$$V^{-1}AV = V^{-1}(A[\vec{v_1} \; \dots \; \vec{v_n}]) =$$

$$= V^{-1}[A\vec{v_1} \; \dots \; A\vec{v_n}]$$

from 16A:

$$A\vec{v_i} = \lambda_i \cdot \vec{v_i}$$

eigenvectors, eigenvalues

$$= V^{-1}[\lambda_1\vec{v_1} \; \dots \; \lambda_n\vec{v_n}]$$

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$$

if $V = [\vec{v_1} \; \dots \; \vec{v_n}]$

$\vec{v_i}$'s are linearly indep and each eigenvectors of A

So, if V is an eigenbasis (basis of eigenvectors) then:

$$V^{-1}AV = \Lambda \quad \text{and}$$

a collection of scalar D.E.

$$\frac{d}{dt}\vec{\tilde{x}}(t) = V^{-1}AV\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t)$$

☺

Want to find eigenvectors and eigenvalues of A, to compute V and Λ.

Use our 2nd order RC circuit matrix:

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Recall:

$$A\vec{v} = \lambda\vec{v}$$

$$\underbrace{(A - \lambda I)}_{\text{has a null-space}}\vec{v} = 0$$

$\det(A - \lambda I) = 0$ if λ is eigenvalue

or $\det(\lambda I - A) = 0$

$\det(\lambda I - A)$:

$$\det\begin{pmatrix} \lambda+5 & -2 \\ -2 & \lambda+2 \end{pmatrix} =$$

$$= (\lambda+5)(\lambda+2) - 4$$

$$= \lambda^2 + 7\lambda + 6$$

$$= (\lambda+6)(\lambda+1) = 0$$

so $\boxed{\lambda_1 = -1, \quad \lambda_2 = -6}$

$$\boxed{\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}}$$

$A - \lambda I$: $\lambda_1 = -1$

$$\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

Null-space for $(A - \lambda_1 I)$? $\rightarrow \vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

$\lambda_2 = -6$

$$\begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \underbrace{V^{-1}AV}_{\Lambda}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t)$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}\vec{\tilde{x}}(t) = \begin{bmatrix} -\tilde{x}_1(t) \\ -6\tilde{x}_2(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \vec{\tilde{x}}_1(t) \\ \vec{\tilde{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -\tilde{x}_1(t) \\ -6\tilde{x}_2(t) \end{bmatrix}$$

$$\frac{d}{dt}\hat{x}_1(t) = -x_1(t) \qquad \frac{d}{dt}\hat{x}_2(t) = -6\hat{x}_2(t)$$

scalar diff. equations

$$\tilde{x}_1(t) = \hat{x}_1(0)\cdot e^{-t} \qquad \tilde{x}_2(t) = \tilde{x}_2(0)\cdot e^{-6t}$$

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \tilde{x}_1(0)\, e^{-t} \\ \tilde{x}_2(0)\, e^{-6t} \end{bmatrix}$$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}(0)} = \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \end{bmatrix}$$

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} \end{bmatrix}$$

but want $x(t)$

$$\vec{x}(t) = V\,\vec{\tilde{x}}(t) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} \frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t} \\ \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t} \end{bmatrix}$$

Summary: $\boxed{\dfrac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)}$

Original coord:

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\
\vec{\tilde{x}} \arrow[r, "V^{-1}AV"'] \arrow[u, bend right, "V"'] & \frac{d}{dt}\vec{\tilde{x}} \arrow[u, bend right, "V"']
\end{tikzcd}$$

Nice coordinates

$\vec{\tilde{x}} = V^{-1}\vec{x}$ $\qquad\qquad \vec{x} = V\vec{\tilde{x}}$

$$\frac{d}{dt}\vec{\tilde{x}} = \frac{d}{dt}V^{-1}\vec{x} = V^{-1}\frac{d}{dt}\vec{x} = V^{-1}\left(A\vec{x} + B\vec{u}\right)$$

$$= V^{-1}AV\vec{\tilde{x}} + V^{-1}B\vec{u}$$

$$\boxed{\frac{d}{dt}\vec{\tilde{x}}(t) = \underbrace{V^{-1}AV}_{}\vec{\tilde{x}} + V^{-1}B\vec{u}}$$

eigenvals $\rightarrow \Lambda$ for V eigenvectr