Lecture 7

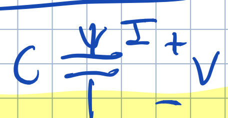
EECS 16B

* Intro to inductors
* 2^{nd} order systems with complex eigenvalues
  complex eigenvalues

Lab-lite sections — just like HW party!

Join us and be done with the lab in 3 hrs!

Capacitor



$$\boxed{I(t) = C \frac{d}{dt} V(t)}$$

capacitor "fights the change in voltage"

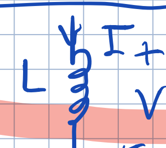
stores energy in electric field

$$\boxed{E = \frac{1}{2} C V^2}$$

Farad [F]

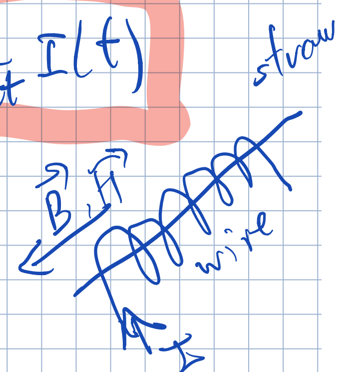
at DC: acts as an open-circuit

Inductor



$$\boxed{V(t) = L \frac{d}{dt} I(t)}$$

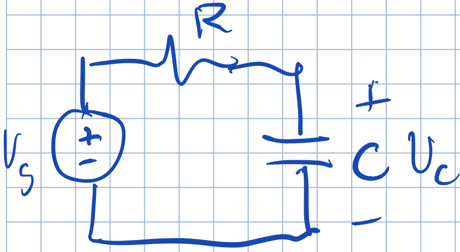
"fights the change in current"



stores energy in magnetic field:

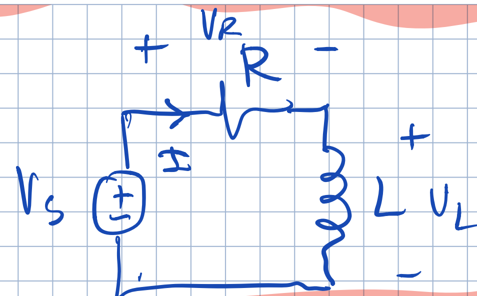
$$\boxed{E = \frac{1}{2} L I^2}$$

short-circuit at constant current



$$V_C \sim e^{-\frac{t}{RC}}$$

$$RC = \tau$$
time constant



Time constant $\tau = ?$

Decaying exp: $e^{-\frac{t}{\tau}}$

$$V_s(t) = 1V \quad , t \leq 0 \qquad 1V$$
$$V_s(t) = 0V \quad , t > 0 \qquad \downarrow 0V$$

At the beginning $V_R(0) = V_s(0) - \cancel{V_L(0)}^{\,0}$

$$= 1V$$

$$I(0) = \frac{V_R(0)}{R} = \frac{1V}{R}$$

$$V_L(t) = L\frac{d}{dt}I(t)$$

$$V_R(t) = R \cdot I(t)$$

$$V_s(t) = V_L(t) + V_R(t)$$

$$V_L(t) = -V_R(t) \quad , V_s(t) = 0, t > 0$$

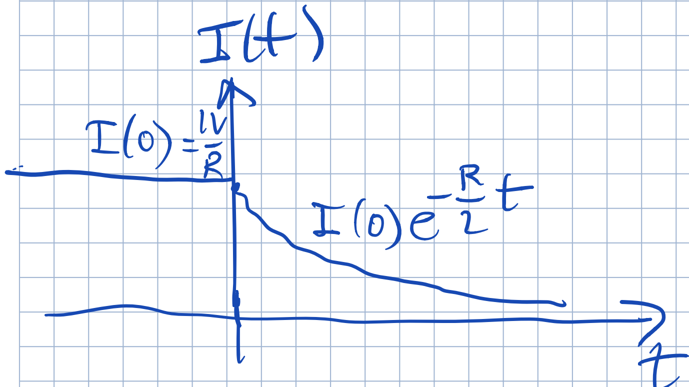
$$L\frac{d}{dt}I(t) = -R\,I(t)$$

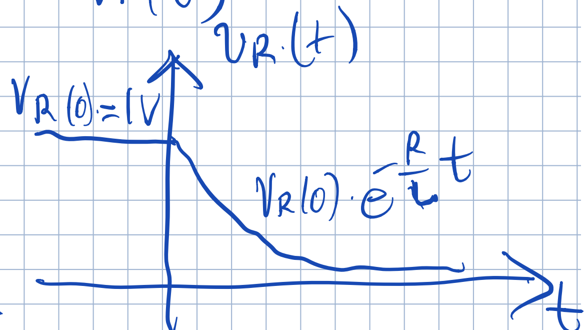
$$\boxed{\frac{d}{dt}I(t) = -\frac{R}{L}I(t)}$$

$$I(t) = I(0)\cdot e^{-\frac{R}{L}t} = I(0)\cdot e^{-\frac{t}{\tau}}$$

time constant $\rightarrow \tau = \frac{L}{R}$

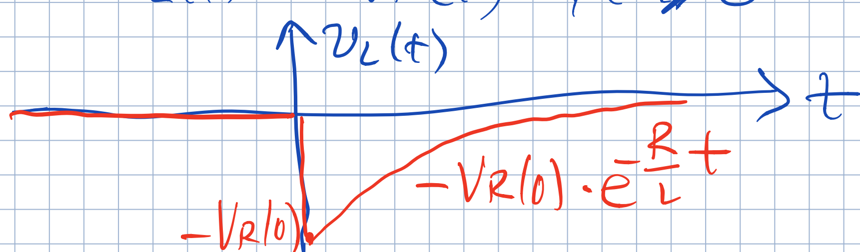
$$V_R(t) = R\,I(t) = \underbrace{R\,I(0)}_{V_R(0)}\,e^{-\frac{t}{\tau}}$$

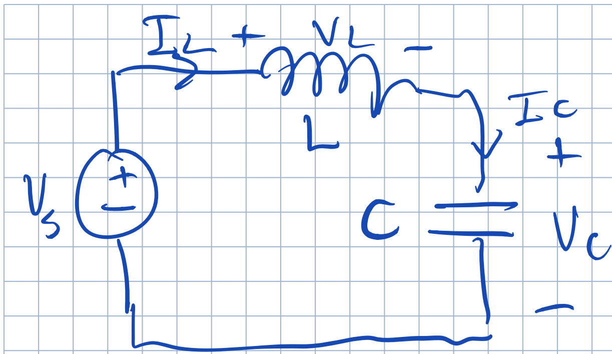




$$v_L(t) = V_S(t) - V_R(t)$$

$$v_L(t) = -V_R(t) \quad , t \geq 0$$





$$I_C(t) = C\frac{d}{dt}v_C(t)$$

$$v_L(t) = L\frac{d}{dt}I_L(t)$$

$$I_L(t) = I_C(t) \quad (KCL)$$

$V_S(t<0) = 1V$

$V_S(t \geq 0) = 0V$

$v_C(0) = 1V$

$I_L(0) = 0A$

$$V_S(t) = v_C(t) + v_L(t) \quad (KVL)$$

$t \geq 0, \ V_S(t) = 0$

$$v_L(t) = -v_C(t)$$

$$v_L(t) = L\frac{d}{dt}I_L(t) = -v_C(t) \Rightarrow \boxed{\frac{d}{dt}I_L(t) = -\frac{1}{L}v_C(t)}$$

$$I_L(t) = I_C(t) = C\frac{d}{dt}v_C(t) \Rightarrow \boxed{\frac{d}{dt}v_C(t) = \frac{1}{C}I_L(t)}$$

$$\frac{d}{dt}\begin{bmatrix} I_L(t) \\ v_C(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} \begin{bmatrix} I_L(t) \\ v_C(t) \end{bmatrix}$$

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

$$\vec{x}(t) = \begin{bmatrix} I_L(t) \\ v_C(t) \end{bmatrix}$$

$\Downarrow$

diagonalize A

to solve

Calculate eigenvalues of A &
eigenvectors of A.

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & \frac{1}{L} \\ -\frac{1}{C} & \lambda \end{pmatrix} = \lambda^2 + \frac{1}{LC} = 0$$

$$\lambda^2 = -\frac{1}{LC}$$

$$\boxed{\lambda_{1,2} = \pm j\sqrt{\frac{1}{LC}}}$$ where $j = \sqrt{-1}$

Have the form: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where

$$\vec{x}(t) = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$$

Assume: $L = 1H$, $C = 1F$ (really large values)

$$\Rightarrow A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm j$$

want to find $\vec{v_1}, \vec{v_2}$ s.t.

$$A\vec{v_1} = \lambda_1 \vec{v_1} \quad \& \quad A\vec{v_2} = \lambda_2 \vec{v_2}$$

Nullspace style

$\lambda_1 = j$

$$(A - \lambda_1 I) \cdot \vec{v_1} = 0$$

$$\begin{bmatrix} -j & -1 \\ 1 & -j \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v_1} = \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

Ad-hoc style

$\lambda_2 = -j$ any multiple of eigenvector is an eigenvector from eigenspace

$$A k \vec{v_2} = \lambda_2 k \vec{v_2}$$

$$\vec{v_2} = \begin{bmatrix} 1 \\ ? \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = -j \begin{bmatrix} 1 \\ x \end{bmatrix}$$

$$-x = -j \Rightarrow x = j$$

$$\vec{v_2} = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$V = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{j}{2} \\ \frac{1}{2} & -\frac{j}{2} \end{bmatrix}$$

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A}\vec{x}(t)$$

$$\vec{x}(t) = V\vec{\tilde{x}}(t)$$

$$\vec{\tilde{x}}(t) = V^{-1}x(t)$$

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \\
\vec{\tilde{x}} \arrow[r, "\underbrace{V^{-1}AV}_{\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}"'] \arrow[u, bend right, "V"'] & \frac{d}{dt}\vec{\tilde{x}}
\end{tikzcd}$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \underbrace{\begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}}_{\Lambda = V^{-1}AV}\vec{\tilde{x}}(t)$$

$$\tilde{x}_1(t) = \tilde{x}_1(0)\, e^{jt}$$

$$\tilde{x}_2(t) = \tilde{x}_2(0)\, e^{-jt}$$

$$I_L(0) = 0A$$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} \frac{1}{2} & \frac{j}{2} \\ \frac{1}{2} & -\frac{j}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{j}{2} \\ -\frac{j}{2} \end{bmatrix}$$

$$u_c(0) = 1V$$

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \frac{j}{2} e^{jt} \\ -\frac{j}{2} e^{-jt} \end{bmatrix}$$

$$\vec{x}(t) = V \cdot \vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \cdot \begin{bmatrix} \frac{j}{2} e^{jt} \\ -\frac{j}{2} e^{-jt} \end{bmatrix} = \begin{bmatrix} \frac{je^{jt} - je^{-jt}}{2} \\ \frac{e^{jt} + e^{-jt}}{2} \end{bmatrix}$$

Use Euler formula (or from Taylor exp):

$$e^{j\theta} = \cos\theta + j\sin\theta$$

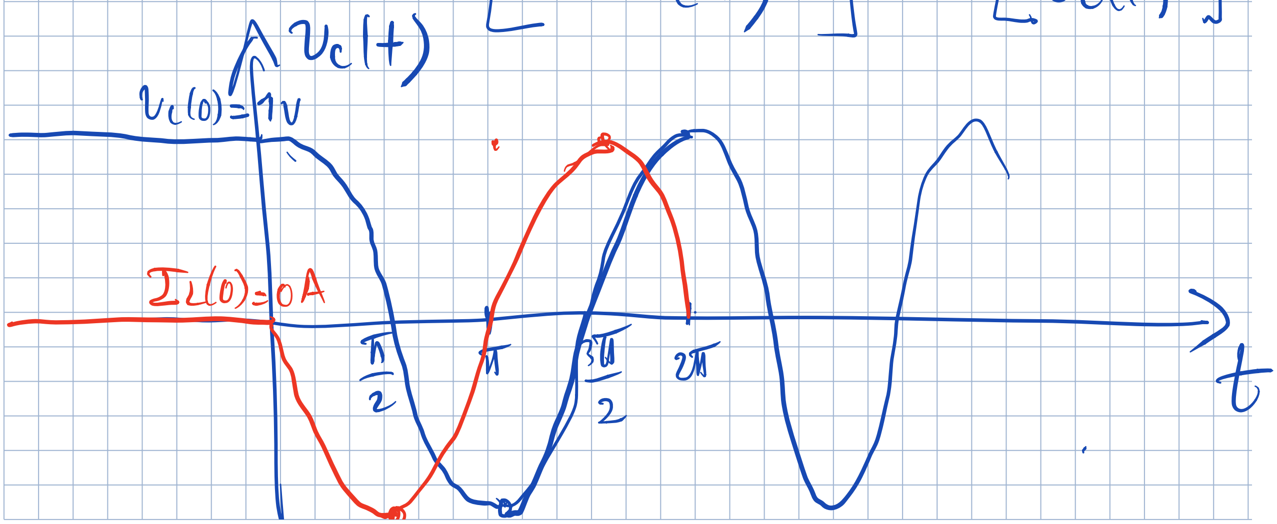
$$\frac{1}{2}\left(j\left(\cos t + j\sin t\right) - j\left(\cos(-t) + j\sin(-t)\right)\right)$$

$$= \frac{1}{2}\left(\cancel{j\cos t} - \sin t \cancel{- j\cos t} + j^2 \sin t\right)$$

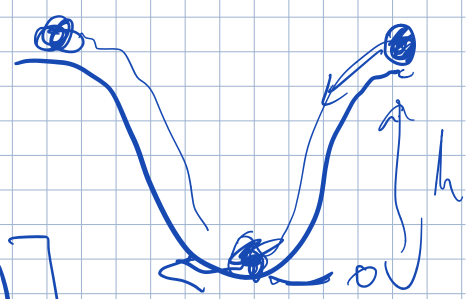
$$= -\frac{2\sin(t)}{2} = \boxed{-\sin(t)}$$

$$\frac{e^{jt} + e^{-jt}}{2} = \cos(t)$$

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ v_C(t) \end{bmatrix}$$



Note: $\lambda_{1,2} = \pm i\sqrt{\frac{1}{LC}}$



For the general case:

$$\vec{x}(t) = \begin{bmatrix} -\sin\left(\sqrt{\frac{1}{LC}}\,t\right) \\ \cos\left(\sqrt{\frac{1}{LC}}\,t\right) \end{bmatrix}$$