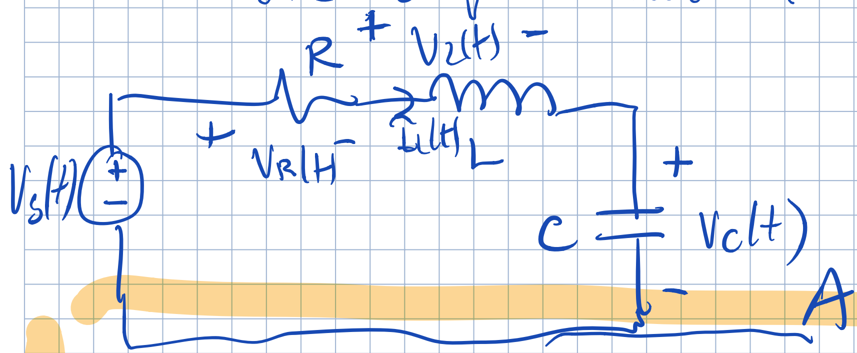
Lecture 8 EECS 16B

* RLC circuits
* Solving systems of diff. eqns. with Phasors

A note on homework:



$$\frac{d}{dt}\begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_s(t)$$

$V_L(t) = V_s(t) - V_R(t) - V_C(t)$: KVL

$$L\frac{d}{dt}I_L(t) = V_s(t) - I_L(t)\cdot R - V_C(t)$$

$$\frac{d}{dt}I_L(t) = -\frac{R}{L}I_L(t) - \frac{1}{L}V_C(t) + \frac{1}{L}V_s(t)$$

How to solve:

1) Find eigenvalues & eigenvectors of A

2) Change the coordinates to $\vec{\tilde{x}}(t)$ in the eigenbasis of A.

3) Solve a simpler problem $\vec{\tilde{x}}(t) \Rightarrow$

- use $\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0)$ ($\hat{A} = V^{-1}AV$ would be diagonal or upper triangular)

$$\vec{\hat{x}}(t) = \begin{bmatrix} x_1(0)\, e^{\lambda_1 t} \\ \vdots \\ x_n(0)\, e^{\lambda_n t} \end{bmatrix}$$

4) Change the coordinates back:

$$\vec{x}(t) = V \cdot \vec{\hat{x}}(t)$$

Reminder:

For $\frac{d}{dt} x(t) = \lambda x(t) + u(t)$

when: $u(t) = K \cdot e^{st}$ when $s \neq \lambda$

$$x(t) = \underbrace{\left(x(0) - \frac{K}{s-\lambda}\right)e^{\lambda t}}_{\text{Annoying}} + \underbrace{\frac{K}{s-\lambda} \cdot e^{st}}_{\text{Nice}}$$

Annoying (comes from initial conditions)

Nice (same form as input) steady-state solution

When can we ignore?

When does $e^{\lambda t} \to 0$?

$e^{\lambda t} \to 0$ as $t \to \infty$ iff $\lambda < 0$

What about complex λ's?

$$e^{\lambda t} = e^{(\lambda r + j\lambda i)t} = \boxed{e^{\lambda r t}} \cdot e^{j\lambda i t} =$$

$$= \boxed{e^{\lambda r t}} \underbrace{(\cos \lambda i t + j \sin \lambda i t)}_{\text{does not go away in } t}$$

$\downarrow 0$ iff $\lambda r < 0$

Let's consider that inputs are of the form $\sim e^{st}$ and assert that all other quantities (solutions) are $\sim e^{st}$, valid if $s \neq \lambda$ and $Re(\lambda) = \lambda_r < 0$.

Let's try: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t)$

when $\boxed{\vec{u}(t) = \vec{\tilde{u}}\, e^{st}}$ $\quad \vec{\tilde{u}}$ vector of constants

Assert: $\vec{x}(t) = \vec{\tilde{x}} \cdot e^{st}$: $\vec{\tilde{x}}$ vector of constants

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}(\vec{\tilde{x}} e^{st}) = \vec{\tilde{x}} \cdot \frac{d}{dt} e^{st} = \vec{\tilde{x}} \cdot s e^{st} =$$

$$= s\vec{\tilde{x}} e^{st} = A\vec{\tilde{x}} e^{st} + \vec{\tilde{u}} e^{st}$$

$$s\vec{\tilde{x}} \cancel{e^{st}} = A\vec{\tilde{x}} \cancel{e^{st}} + \vec{\tilde{u}} \cancel{e^{st}}$$

$$s\vec{\tilde{x}} = A\vec{\tilde{x}} + \vec{\tilde{u}}$$

$$\boxed{(sI-A)\vec{\tilde{x}} = \vec{\hat{u}}}$$

System of linear eqns.

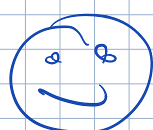
$$\boxed{\vec{\tilde{x}} = (sI-A)^{-1}\cdot\vec{\hat{u}}}$$

Remember $s \neq \lambda$ so $sI-A$ has no nullspace & is therefore invertible!

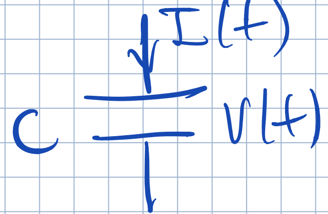
$$\vec{x}(t) = \vec{\tilde{x}}\, e^{st}$$

$$\boxed{\vec{x}(t) = (sI-A)^{-1}\,\vec{\hat{u}}\, e^{st}}$$

We solved system of diff. equations by solving a system of linear eqns.!



Can we use this for ckts directly?



$$I(t) = C\frac{d}{dt}V(t)$$

$$I(t) = \tilde{I}e^{st}$$

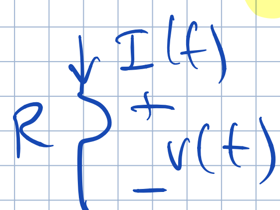
$$V(t) = \hat{V}e^{st} \qquad \frac{V(t)}{I(t)} = \frac{\hat{V}}{\tilde{I}}$$

$$I(t) = \tilde{I}e^{st} = C\frac{d}{dt}\left(\hat{V}e^{st}\right) = C\tilde{V}\underbrace{\frac{d}{dt}e^{st}}_{se^{st}}$$

$$\tilde{I}\cancel{e^{st}} = sC\hat{V}\cancel{e^{st}}$$

$$\boxed{\frac{\tilde{V}}{\tilde{I}} = \frac{1}{sC}} \leftarrow$$

capacitor s-impedance



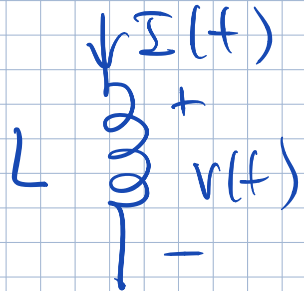
$$v(t) = \hat{v}e^{st}$$

$$I(t) = \hat{I}e^{st}$$

$$\underbrace{\hat{V}\cancel{e^{st}}}_{v(t)} = \underbrace{\tilde{I}\cancel{e^{st}}}_{I(t)} \cdot R$$

$$\boxed{\frac{\tilde{V}}{\tilde{I}} = R} \leftarrow$$

resistor s-impedance



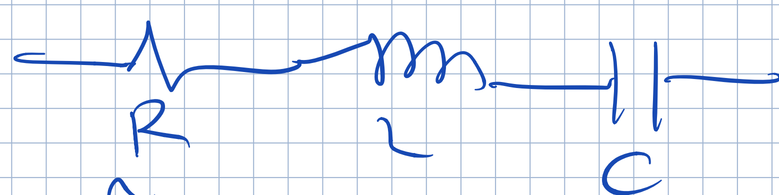
$$v(t) = L\frac{d}{dt}I(t)$$

$$\left.\begin{aligned}\hat{v}(t) &= \hat{v}e^{st}\\ \hat{I}(t) &= \hat{I}e^{st}\end{aligned}\right\}$$

$$\hat{V}e^{st} = L\frac{d}{dt}(\hat{I}e^{st}) = L\hat{I}\underbrace{\frac{d}{dt}e^{st}}_{s\cdot e^{st}}$$

$$\hat{V}\cancel{e^{st}} = sL\hat{I}\cancel{e^{st}}$$

$$\boxed{\frac{\hat{V}}{\hat{I}} = sL}$$ ⇐ inducta s-impedance



$Z_R = R$ $\quad Z_L = sL$ $\quad Z_c = \frac{1}{sC}$

$\hat{V}, \hat{I}$

$\hat{V}_R = \hat{I}\cdot Z_R$

$\hat{V}_L = \hat{I}\cdot Z_L$

$\hat{V}_c = \hat{I}\cdot Z_c$

Summary:

sys. diff. eqns.

$$\boxed{\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t)}$$

& $\vec{u}(t) = \hat{u}\, e^{st}$, $s \neq \lambda$

we have: $\vec{x}(t) = \tilde{\vec{x}}\, e^{st}$

& get

$$\boxed{s\tilde{\vec{x}}(t) = A\tilde{\vec{x}} + \hat{\vec{u}}}$$

system of lin equations.

$$\tilde{\vec{x}}(t) = (sI - A)^{-1}\hat{\vec{u}}$$

(steady-state solution)

solutions to:

$$\frac{d}{dt}\vec{x}(t) - A\vec{x}(t) = 0$$

can always be added to steady-state solution.

– of the form $\vec{x}_h(t) = \vec{x}(0)\, e^{\lambda t}$

where λ is eigenvalue of A.

(these come from init. conditions)

To ignore: $\lambda_r < 0$ can focus only on steady-state solution.

For sinusoidal inputs:

$$u(t) = U\cos(\omega t + \phi) = U \cdot \frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2}$$

$$u(t) = \underbrace{\frac{U \cdot e^{j\phi}}{2}}_{\tilde{u}} \cdot \underbrace{e^{j\omega t}}_{e^{s_1 t}} + \underbrace{\frac{U \cdot e^{-j\phi}}{2}}_{\bar{\tilde{u}}} \cdot \underbrace{e^{-j\omega t}}_{e^{s_2 t}}$$

$s_1 = j\omega$ $\qquad$ $s_2 = -j\omega$

always complex conjugates

$$u(t) = \tilde{u}\, e^{j\omega t} + \hat{u} \cdot e^{-j\omega t}$$

Use superposition to solve for ckts V's & I's:

For $s_1 = j\omega$

$$M = j\omega I - A = s_1 I - A$$

Topology of the ckt elements $\rightarrow$ $M \begin{bmatrix} \vec{\tilde{I}} \\ \vec{\tilde{V}} \end{bmatrix} = \vec{\tilde{u}}$ $\leftarrow$ indep sources

solve: $\boxed{\begin{bmatrix} \vec{\tilde{I}} \\ \vec{\tilde{V}} \end{bmatrix} = M^{-1} \vec{\tilde{u}}}$

$s_2 = -j\omega$ get:

$$\bar{M}\begin{bmatrix}\vec{\tilde{I}} \\ \vec{\tilde{V}}\end{bmatrix} = \bar{\vec{\tilde{u}}}$$

$\bar{M} = s_2 I - A$
$= -j\omega I - A$

since:

$\overline{M\vec{x}} = \bar{M}\cdot\bar{\vec{x}}$

$$\overline{\begin{bmatrix}\vec{\tilde{I}} \\ \vec{\tilde{V}}\end{bmatrix}} = \begin{bmatrix}\bar{\vec{\tilde{I}}} \\ \bar{\vec{\tilde{V}}}\end{bmatrix} = \bar{M}^{-1}\cdot\bar{\vec{\tilde{u}}}$$

$$= \underbrace{\overline{M^{-1}\vec{\tilde{u}}}}$$

all solutions:

$$\vec{V}(t) = \boxed{\vec{\tilde{V}}}\, e^{j\omega t} + \bar{\vec{\tilde{V}}}\cdot e^{-j\omega t}$$

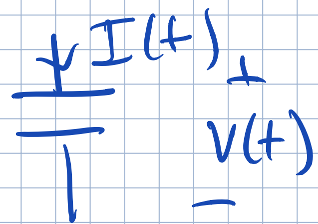
$$\vec{I}(t) = \boxed{\vec{\tilde{I}}}\, e^{j\omega t} + \bar{\vec{\tilde{I}}}\cdot e^{-j\omega t}$$

$s = j\omega$:

$\hat{V}$ & $\hat{I}$ are phasors (functions of ω)

s-impedances are called impedances (functions of ω)

Phasors reduce 16B to 16A problems!



$$v(t) = V_0 \cos(\omega t + \phi)$$

$$v(t) = \underbrace{\frac{V_0}{2} e^{j\phi}}_{\hat{V}} \cdot e^{j\omega t} + \underbrace{\frac{V_0}{2} e^{-j\phi}}_{\bar{\hat{V}}} e^{-j\omega t}$$

$$I(t) = C \frac{d}{dt} v(t) =$$

$$= C \frac{d}{dt} \left(\hat{V} e^{j\omega t} + \bar{\hat{V}} e^{-j\omega t} \right)$$

$$= \underbrace{j\omega C \hat{V}}_{\hat{I}} e^{j\omega t} \underbrace{- j\omega C \bar{\hat{V}}}_{\bar{\hat{I}}} e^{-j\omega t}$$

$$I(t) = \hat{I} e^{j\omega t} + \bar{\hat{I}} e^{-j\omega t}$$

$$\hat{I} = j\omega C \hat{V} \Rightarrow Z_c = \frac{\hat{V}}{\hat{I}} = \boxed{j\omega}$$