EECS 16B — Module 2, Lecture 5

March 11, 2021.

- Midterm

Last time: Controllability

Today: Machine Learning / Robotics.

- Handling model changes / unknown models.
  - ↳ Repeated Least squares.
- Gram-Schmidt Orthonormalization.
  - ↳ Projections.

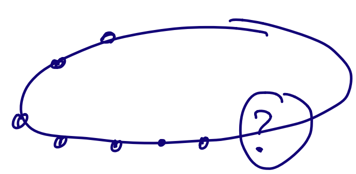
Recap: Least Squares. $A\vec{x} \approx \vec{b}$

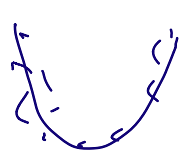
$$\underbrace{\begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

Unknowns: $x_1, x_2 \cdots x_n$.
weights on columns of A: $\vec{a_1}, \vec{a_2} \cdots \vec{a_n}$

16A: ① Imaging Lab : unknowns: pixels.

② Polynomial fitting:





$\alpha x^2 + \beta x + \gamma = 0$

Unknowns: α, β, γ

③ System Identification. $x[t+1] = a x[t] + b u[t]$

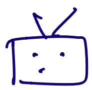
Polynomial fitting:

$\alpha x^2 + \beta x + \gamma = 0$ Choice 1

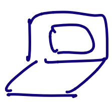
$\alpha x^3 + \beta x^2 + \gamma x + \delta = 0$ choice 2

Example for today:

"Internet of Things"

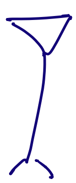


DI)) $\vec{s_1}$

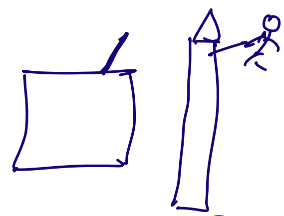


)) $\vec{s_2}$

)) $\vec{s_3}$



)) $\vec{s_4}$



Air traffic Controller

$\vec{r}$

$\vec{r} = \alpha_1 \vec{s_1} + \alpha_2 \vec{s_2} + \cdots$

I want to know:

Who is transmitting?

$\vec{s_1}, \vec{s_2}, \vec{s_3}, \vec{s_4}$

$$\begin{bmatrix} \vec{s_1} & \vec{s_2} & \vec{s_3} & \vec{s_4} \\ & & & \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \vec{r} \\ \\ \end{bmatrix}$$

What if I only find out about each of the devices one by one?

Initially: $\vec{s_1}$, $\vec{r}$

- Does $\vec{s_1}$ explain $\vec{r}$?

LS: $\begin{bmatrix} \vec{s_1} \end{bmatrix} [\alpha_1] = \begin{bmatrix} \vec{r} \end{bmatrix}$

Find α_1 using least squares.

↳ get some estimate.

Expanded set: $\{\vec{s_1}, \vec{s_2}\}$.

LS: $\begin{bmatrix} \vec{s_1} & \vec{s_2} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \vec{r} \end{bmatrix}$

New set: $\{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$

LS: $\underbrace{\begin{bmatrix} \vec{s_1} & \vec{s_2} & \vec{s_3} \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \vec{r} \end{bmatrix}$

Observe: A keeps getting bigger

$A^TA \longrightarrow$ more work.

$(A^TA)^{-1} \longrightarrow$ more work.

Properties of orthonormality.

Least-Squares is easy is the columns of matrix A are orthonormal.

$$\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \end{bmatrix}$$

$\|\vec{q}_i\| = 1 = \langle \vec{q}_i, \vec{q}_i \rangle$ Normalized

$\langle \vec{q}_i, \vec{q}_j \rangle = 0 \quad \forall i, j, i \neq j$

Orthogonal.

$A\vec{x} = \vec{b}$

$A = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$ Orthonormal.

$$\hat{\vec{x}} = (A^T A)^{-1} A^T \cdot \vec{b}$$

$$= \left(\begin{bmatrix} -\vec{q}_1^{\,T}- \\ -\vec{q}_2^{\,T}- \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \right)^{-1} A^T \cdot \vec{b}$$

$$= \left(\begin{bmatrix} \vec{q}_1^{\,T} q_1 & \vec{q}_1^{\,T} \vec{q}_2 \\ \vec{q}_2^{\,T} \vec{q}_1 & \vec{q}_2^{\,T} \vec{q}_2 \end{bmatrix} \right)^{-1} A^T \vec{b}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\vec{q}_1^{\,T}- \\ -\vec{q}_2^{\,T}- \end{bmatrix} \vec{b}$$

$$= \boxed{\begin{bmatrix} \langle \vec{q}_1, \vec{b} \rangle \\ \langle \vec{q}_2, \vec{b} \rangle \end{bmatrix}}$$

$$A = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} \longrightarrow \text{Orthonormal.} \qquad A\vec{x} = \vec{b}$$

$$\underline{A^T A} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\vec{x}} = \underbrace{(A^T A)^{-1}} A^T \vec{b}$$

$$= \begin{bmatrix} - \vec{q}_1^{\,T} - \\ - \vec{q}_2^{\,T} - \\ - \vec{q}_3^{\,T} - \end{bmatrix} \begin{bmatrix} \vec{b} \end{bmatrix}$$

$$= \begin{bmatrix} \langle \vec{q}_1, \vec{b} \rangle \\ \langle \vec{q}_2, \vec{b} \rangle \\ \langle \vec{q}_3, \vec{b} \rangle \end{bmatrix}$$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \vec{y}^T \vec{x}$$

We can reuse previous work! Yay!

<u>Goal</u>: To find $\vec{r}$ as a linear combination of $\vec{s}_1, \vec{s}_2, \vec{s}_3 \ldots$, where we only find out the $\vec{s}_i$ vectors one at a time.

$\vec{s}_1, \vec{s}_2, \vec{s}_3 \longrightarrow$ Convert them to $\vec{q}_1, \vec{q}_2, \vec{q}_3$

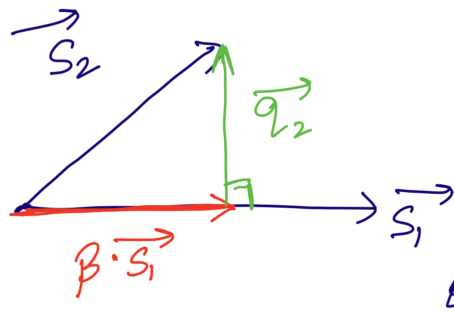
such that $\vec{q}_1, \vec{q}_2, \vec{q}_3$ etc. are all orthonormal.

Focus now: How to orthogonalize signatures as they come in.

$\{\vec{s_1}, \vec{s_2}\}$.

$\vec{s_2}$ in terms of $\vec{s_1}$

$$\vec{s_2} = \underbrace{\beta \cdot \vec{s_1}}_{\text{aligned to } \vec{s_1}} + \underbrace{\vec{q_2}}_{\text{brand "new"}}$$



$$\vec{s_2} = \underline{\underline{\beta}} \cdot \vec{s_1} + \vec{q_2}$$

$$\vec{q_2} = \vec{s_2} - \beta \vec{s_1}$$

Review of projections

$$\langle \vec{s_1}, \vec{q_2} \rangle = 0$$

$$\Rightarrow \langle \vec{s_1}, \vec{s_2} - \beta \vec{s_1} \rangle = 0$$

$$\Rightarrow \langle \vec{s_1}, \vec{s_2} \rangle - \beta \cdot \|\vec{s_1}\|^2 = 0$$

$$\Rightarrow \quad \beta = \frac{\langle \vec{s_1}, \vec{s_2} \rangle}{\|\vec{s_1}\|^2}$$

Gram-Schmidt Algorithm / Procedure

Given $\{\vec{s_1}, \vec{s_2}, \dots \vec{s_n}\}$

Convert this to a set

$$\{\vec{q_1}, \vec{q_2}, \dots \vec{q_n}\}.$$

such that: $\langle \vec{q_i}, \vec{q_i} \rangle = \|\vec{q_i}\|^2 = 1$

$$\langle \vec{q_i}, \vec{q_j} \rangle = 0$$

and:

$$\text{span}\{\vec{s_1}\} = \text{span}\{\vec{q_1}\}$$

$$\text{span}\{\vec{s_1}, \vec{s_2}\} = \text{span}\{\vec{q_1}, \vec{q_2}\}.$$

$$\vdots$$

$$\text{span}\{\vec{s_1}, \vec{s_2}, \dots \vec{s_n}\} = \text{span}\{\vec{q_1}, \vec{q_2}, \dots \vec{q_n}\}.$$

Consider: linearly independent set.

$\{\vec{s_1}, \vec{s_2}, \dots \vec{s_n}\}$ lin indep.

Gram-Schmidt - Alg.

$\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$.

① $\{\vec{s}_1\} \longrightarrow \vec{q}_1$ find.

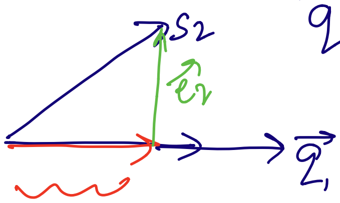
$$\frac{\vec{s}_1}{\|\vec{s}_1\|} = \vec{q}_1 \longrightarrow \text{unit norm.}$$

$$\text{span}\{\vec{q}_1\} = \text{span}\{\vec{s}_1\} \checkmark$$

② $\{\vec{s}_1, \vec{s}_2\}$ $\boxed{\vec{q}_1}$

What is new in $\vec{s}_2$, that is not captured by $\vec{q}_1$.

Remove from $\vec{s}_2$, the projection of $\vec{s}_2$ onto $\vec{q}_1$



$$\vec{e}_2 = \vec{s}_2 - \frac{\langle \vec{s}_2, \vec{q}_1 \rangle}{\|\vec{q}_1\|^2} \cdot \vec{q}_1$$

$$\vec{e}_2 = \vec{s}_2 - \langle \vec{s}_2, \vec{q}_1 \rangle \cdot \vec{q}_1$$

$$\vec{q}_2 = \frac{\vec{e}_2}{\|\vec{e}_2\|} \longrightarrow \underline{\underline{\text{unit norm}}}$$

<u>Check</u>: $\langle \vec{q}_2, \vec{q}_1 \rangle = \left\langle \frac{\vec{e}_2}{\|\vec{e}_2\|}, \vec{q}_1 \right\rangle$

$$= \left\langle \frac{\vec{s}_2 - \langle \vec{s}_2, \vec{q}_1 \rangle \cdot \vec{q}_1}{\|\vec{e}_2\|}, \vec{q}_1 \right\rangle$$

$$= \frac{1}{\|\vec{e}_2\|}\left[\langle \vec{s}_2, \vec{q}_1 \rangle - \langle \vec{s}_2, \vec{q}_1 \rangle \underbrace{\langle \vec{q}_1, \vec{q}_1 \rangle}_{1} \right]$$

$$= 0$$

<u>Furthermore:</u> $\text{span}\{\vec{s}_1, \vec{s}_2\} = \text{span}\{\vec{q}_1, \vec{q}_2\}$

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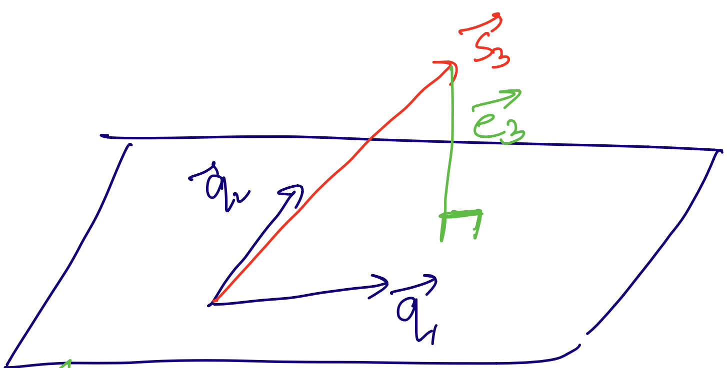
(3) $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$. $\boxed{\vec{q}_1, \vec{q}_2}$



Project $\vec{s}_3$ onto span $\{\vec{q}_1, \vec{q}_2\}$

$$\text{proj} = \begin{bmatrix} \langle \vec{s}_3, \vec{q}_1 \rangle \\ \langle \vec{s}_3, \vec{q}_2 \rangle \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$$

$$= \underbrace{\langle \vec{s}_3, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{s}_3, \vec{q}_2 \rangle \vec{q}_2}_{\text{true by projection formula and because } \vec{q}_1 \text{ and } \vec{q}_2 \text{ are orthogonal.}}$$

$$\vec{e_3} = \vec{s_3} - \left(\langle \vec{s_3}, \vec{q_1} \rangle \vec{q_1} + \langle \vec{s_3}, \vec{q_2} \rangle \vec{q_2} \right)$$

$$\vec{q_3} = \frac{\vec{e_3}}{\|\vec{e_3}\|} \quad \checkmark$$

Check: $\text{span}\{\vec{q_1}, \vec{q_2}, \vec{q_3}\} = \text{span}\{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$

Check: $\langle \vec{q_3}, \vec{q_1} \rangle = \langle \vec{q_3}, \vec{q_2} \rangle = 0.$

$$\underbrace{\overbrace{\begin{bmatrix} | & | & | \\ \vec{s_1} & \vec{s_2} & \vec{s_3} \\ | & | & | \end{bmatrix}}^{A}}_{\text{Columnspace}} \longleftrightarrow \underbrace{\begin{bmatrix} | & | & | \\ \vec{q_1} & \vec{q_2} & \vec{q_3} \\ | & | & | \end{bmatrix}}_{\text{Basis for columnspace of } A.}$$