EECS 16B Module 2, Lecture 6

Announcements

- Congrats! MT done!
- Short HW. Redo due after Spring Break.
- EECS Womens History Month – lunch after class today.

Last time:

- Gram-Schmidt Orthogonalization

Today:

- Gram Schmidt w/ linearly dependent vectors.
- Properties of orthonormal matrices
- Upper-Triangularization (What to do when a matrix is not diagonalizable?)

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1. $\{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$. Linear dependance

$\vec{s_2} = 2 \cdot \vec{s_1}$

$\rightarrow \{\vec{q_1}, \vec{q_2}, \vec{q_3}\}$

Step: 1 $\vec{s_1} \rightarrow$ normalize

$$\vec{q_1} = \frac{\vec{s_1}}{\|\vec{s_1}\|}$$

Step 2: $\vec{e_2} = \vec{s_2} - \text{proj}_{\vec{q_1}}(\vec{s_2})$

$= \vec{s_2} - \langle \vec{s_2}, \vec{q_1} \rangle \cdot \vec{q_1}$

because $\vec{q_1}$ is unit vector.

$$= \vec{s_2} - \|\vec{s_2}\| \cdot \vec{q_1}$$

$\vec{s_2} = \|\vec{s_2}\| \cdot \vec{q_1}$

$$= \vec{s_2} - \vec{s_2}$$

$$= 0$$

→ Nothing new in $\vec{s_2}$.
Ignore it. Don't add a $\vec{q_2}$ corresponding to $\vec{s_2}$.

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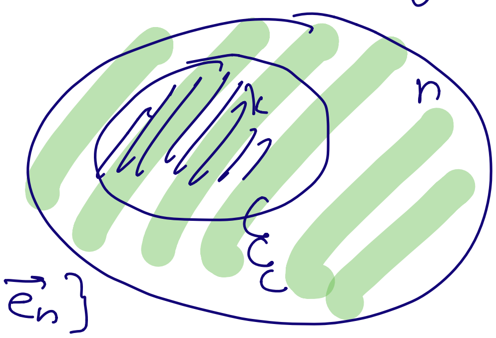
② Gram-Schmidt for building a basis.

$\underbrace{\vec{s_1}, \vec{s_2}, \dots \vec{s_k}}$ $\in \mathbb{R}^n$ $\quad k < n.$

↳ not a basis

Can I create an orthonormal basis using $\vec{s_1}, \vec{s_2} \dots \vec{s_k}$?



→ Consider:

$\{\vec{s_1}, \vec{s_2}, \dots \vec{s_k}, \vec{e_1}, \vec{e_2} \dots \vec{e_n}\}$

$\vec{e_1}, \vec{e_2} \dots \vec{e_n}$ are elementary vector.

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{e_i} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{ith position}$$

→ Do Gram Schmidt in this specific order!

$$\frac{\vec{s_1}}{\|\vec{s_1}\|} = \vec{q_1}, \quad \ldots$$

Max # of lin indep. vectors that can come out : n.

$$\underbrace{\{\vec{q_1}, \vec{q_2} \ldots \vec{q_n}\}}$$ → Guaranteed that all other vectors are lin. combinations

Imp: First ~~vector~~ $\vec{q_1}$ is a scalar multiple of $\vec{s_1}$

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③ Orthonormal matrix R.

$$R = \underbrace{\begin{bmatrix} \vec{q_1} & \vec{q_2} & \vec{q_3} \end{bmatrix}}_{n \times 3 \text{ matrix.}}$$ → orthonormal

$q_i \in \mathbb{R}^n$

$$R^T \cdot R = \underbrace{\begin{bmatrix} - \vec{q_1}^T - \\ - \vec{q_2}^T - \\ - \vec{q_3}^T - \end{bmatrix}}_{3 \times n} \underbrace{\begin{bmatrix} | & | & | \\ \vec{q_1} & \vec{q_2} & \vec{q_3} \\ | & | & | \end{bmatrix}}_{n \times 3}$$

0

$$= \begin{bmatrix} \vec{q_1}^T \vec{q_1} & \vec{q_1}^T \cdot \vec{q_2} & 0 \\ \cdot & \cdot & \cdot \\ & & \cdot \end{bmatrix}_{3\times 3}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

If $R = \begin{bmatrix} \vec{q_1} & \vec{q_2} & \cdots & \vec{q_n} \end{bmatrix}_{n\times n}$ square

$R^T = R^{-1}$ $\qquad$ R: orthonormal !

BIBO stability

$$\vec{x}[k+1] = A\,\vec{x}[k] + \vec{u}[k]$$

Diagonalizable:

if all e-values A : $|\lambda| < 1$

$\longrightarrow$ system is BIBO stable.

→ What if A is not diagonalizable?

→ How do we handle this?

Next-best matrix : Upper-triangular matrix.

RLC

By hook or crook → Upper-triangular.

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \vec{x}(t)$$

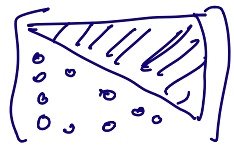
How can we use a basis transformation to convert a matrix into upper-triangular form.

Upper triangularization

Matrix M:

If M diagonalizable: → uses V to transform to diagonal matrix Λ.

If M is not diagonalizable, can we find some transformation U such that

$$U^{-1} M U = \underbrace{T}_{\text{Upp. Tri.}} =$$ 

$$U = \begin{bmatrix} \vec{u_1} & \vec{u_2} & \cdots & \vec{u_n} \end{bmatrix} \qquad \vec{u_1} \ldots \vec{u_n} \text{ form a basis.}$$

<u>Simplest</u> case: M : 1×1 matrix

$M = [m]$ ✓ Upper triangular.

M : 2×2 case.

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

Assume M has all real eigenvalues.

$U = [\ ? \quad ?\ ]$ Try eigenvectors? maybe?

(1) Say $\vec{v_1}$ is one eigenvector of M.

$$M\vec{v_1} = \lambda_1 \vec{v_1} \qquad \vec{v_1} \in \mathbb{R}^2$$

$\hookrightarrow$ Assume $\|\vec{v_1}\| = 1$

$$U = \left[\vec{v}_1 \quad \right]$$

(2) Build out the "basis" using Gram-Schmidt.

Say: $\vec{r}_1$ is the vector you get

$[\vec{v}_1 \quad \vec{r}_1]$ forms a basis

$$\underset{\text{orthogonal}}{\vec{v}_1 \perp \vec{r}_1} \Rightarrow \langle \vec{v}_1, \vec{r}_1 \rangle = 0$$

To check if U works-

try: $U^{-1} M U$

Recall U: orthonormal matrix by construction.

$$= \underbrace{[\vec{v}_1 \quad \vec{r}_1]^{-1}}_{2\times2} \underbrace{M}_{2\times2} \cdot \underbrace{[\vec{v}_1 \quad \vec{r}_1]}_{2\times2}$$

$$= \begin{bmatrix} -\vec{v}_1^{\,T}- \\ -\vec{r}_1^{\,T}- \end{bmatrix} M \, [\vec{v}_1 \quad \vec{r}_1]$$

$$= \begin{bmatrix} \vec{\omega}_1^{\,T} \\ \vec{r}_1^{\,T} \end{bmatrix} \begin{bmatrix} M\vec{\omega}_1 & M\vec{r}_1 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\omega}_1^{\,T} M \vec{\omega}_1 & \vec{\omega}_1^{\,T} M \vec{r}_1 \\ \vec{r}_1^{\,T} M \vec{\omega}_1 & \vec{r}_1^{\,T} M \vec{r}_1 \end{bmatrix}$$

<u>Check:</u> $\vec{r}_1^{\,T} M \vec{\omega}_1 = \vec{r}_1^{\,T} \lambda_1 \vec{\omega}_1 = 0$ ✓

$\underset{\sim}{\vec{\omega}_1^{\,T} M \vec{\omega}_1} = \vec{\omega}_1^{\,T} \cdot \lambda_1 \vec{\omega}_1 = \lambda$

$$= \begin{bmatrix} \lambda_1 & \vec{\omega}_1^{\,T} M \vec{r}_1 \\ 0 & \vec{r}_1^{\,T} M \vec{r}_1 \end{bmatrix} = T$$

$$U^T M U = T$$

<u>Eigenvalues of T:</u>

$$(T - \lambda I) = \begin{bmatrix} \lambda_1 - \lambda & \vec{\omega}_1^{\,T} M \vec{r} \\ 0 & (\vec{r}^{\,T} M \vec{r}_1 - \lambda) \end{bmatrix}$$

$$\underbrace{\det\left((T - \lambda I) \right)}_{\text{Roots of this are e-val of } T} = (\lambda_1 - \lambda)(\vec{r_1}^T M \vec{r_1} - \lambda)$$

$$\Rightarrow \text{evals of } T: \ \underline{\lambda_1}, \ \underline{\vec{r_1}^T M \vec{r_1}}$$

λ_1 is also e-val of M.

Are all e-values of M and T the same?

$M = U^{-1} T U$

$I = U^{-1} U$

$$\det(M - \lambda I)$$

$$= \det(U^{-1} T U - \lambda \cdot U^{-1} U)$$

$$= \det(U^{-1}(T - \lambda I) U)$$

$$= \cancel{\det(U^{-1})} \det(T - \lambda I) \cdot \cancel{\det(U)}$$

$$= \det(T - \lambda I).$$

$\det(AB) = \det(A)\det(B)$

$$\det(U^{-1}) = \frac{1}{\det(U)}$$

Every root of $\det(T-\lambda I)=0$

is also a root of $\det(M-\lambda I)=0$

Build up from the 2×2 case

Case: M: 3×3

$$U^{-1} M U = T.$$

$$U = \begin{bmatrix} \vec{u}_1 & \_ & \_ \end{bmatrix}$$

(1) $M\vec{u}_1 = \lambda_1 \vec{u}_1$ $\quad$ $\vec{u}_1$ is an eigenvector.

(2) Use $\vec{u}_1$ to fill out a basis.

$\{\vec{u}_1, \vec{r}_1, \vec{r}_2\}$ is a orth. normal basis

generated by Gram-Schmidt

Define: $[\vec{r}_1 \ \vec{r}_2] = R$.

$$\underset{3\times 3}{U} = \begin{bmatrix} \underbrace{\vec{u}_1}_{3\times 1} & \underbrace{R}_{3\times 2} \end{bmatrix} \leftarrow \underline{\underline{\text{orthonormal?}}}$$

Consider:

$$\begin{bmatrix} \vec{u}_1 & R \end{bmatrix}^{-1} M \begin{bmatrix} \vec{u}_1 & R \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1^{\,T} \\ R^T \end{bmatrix} M \begin{bmatrix} \vec{u}_1 & R \end{bmatrix}$$

$$\vdots$$

$$= \begin{bmatrix} \lambda_1 & \overset{1\times 2}{\boxed{\vec{u}_1^{\,T} M R}} \\ \boxed{0_{2\times 1}} & \boxed{\underset{2\times 2}{R^T M R}} \end{bmatrix}$$

Recursion?