EECS 16B Designing Information Devices and Systems II
Spring 2021

Note 3B: Inductors and RLC Circuits

1 Inductors

Let's introduce a new passive component, an inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.

1.1 Basics

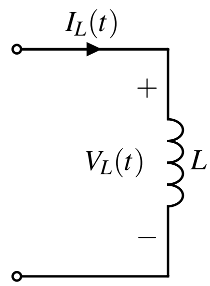


Figure 1: Example Inductor Circuit

The voltage across the inductor is related to its current as follows:

$$V_L(t) = L\frac{dI_L(t)}{dt}. \tag{1}$$

where L is the *inductance* of the inductor. The SI unit of inductance is Henry (H). Looking at eq. (1) we can observe that an inductor behaves as a capacitor with the roles of current and voltage reversed.

Concept Check: The current across the inductor cannot change instantaneously. Why?

Solution: If our current changes instantaneously, then $\frac{d}{dt}I_L \to \infty$, and from eq. (1) the voltage across the inductor $V_L \to \infty$, which is not possible. Hence, our current cannot change instantaneously.

In steady state, when the current flowing through an inductor is constant, there is no voltage drop across the inductor. This makes sense, since an inductor is essentially a spool of wire wrapped around a conductor. Similarly, if the current through the inductor is changing, there will be a voltage drop across the inductor. The energy stored in the inductor turns out to be $E_L = \frac{1}{2}LI^2$, but we won't be using this very much in EECS16B. We are only mentioning it here because it helps us interpret what is happening later.

1.2 Physics behind Inductors
(not in scope for EECS 16B, just for information)

Inductors store energy in a magnetic field. In the same way that a capacitor separates charge (Q) and this leads to an electric field ($\vec{E}$), anytime current flows down a conductor, it creates a magnetic field ($\vec{B}$). Likewise, the magnetic field can store energy. Their behavior can be described using **Faraday's Law of Induction**.

The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries if we want to create a useful inductances. A **solenoid** is a good example, where we wind a wire around a conductor like a copper rod:

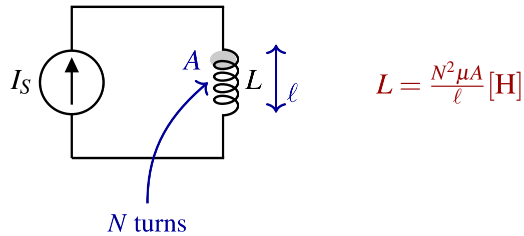


Figure 2: The Inductance of a Solenoid: a wire coiled around something.

Note that the inductance (L) depends on **geometry** and a material property called **magnetic permeability** (μ) of the solenoid core material. In the case of the solenoid in 2, the inductance depends on the number of turns (N), the length of the solenoid (l) and the area (A) of the loops. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc. But in many high speed applications, their presence might be undesirable as they create delays in the time response of the circuit.

1.3 Equivalence Relations

Now that we have the basics, let's derive the equivalence relations for series and parallel combinations of inductors. We will find that these are similar to those of resistors. Why? Because the law governing an inductor $V_L = L\frac{d}{dt}I_L$ involves a proportionality constant L that multiplies a current-like quantity to give a voltage. In a resistor, the resistance R multiplies current to give a voltage.

1.3.1 Series Equivalence

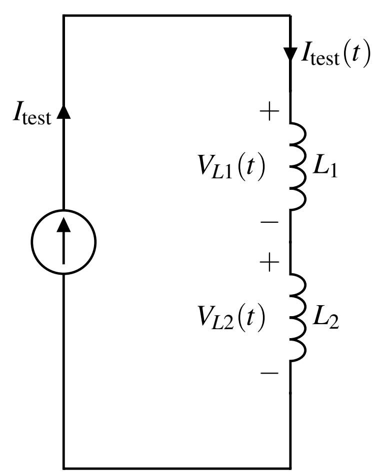


Figure 3: Series Inductor Circuit

Let's apply a $\frac{dI_{\text{test}}}{dt}$ through the two inductors, then

$$V_{L1}(t) + V_{L2}(t) = V_L(t)$$

where, $V_L(t)$ is the voltage across the two inductors. From VI relationship for inductors, we get

$$L_1 \frac{dI_{\text{test}}}{dt} + L_2 \frac{dI_{\text{test}}}{dt} = V_L(t) \tag{2}$$

$$(L_1 + L_2)\frac{dI_{\text{test}}}{dt} = V_L(t) \tag{3}$$

$$L_{\text{eq}} \frac{dI_{\text{test}}}{dt} = V_L(t) \tag{4}$$

where, $L_{\text{eq}} = L_1 + L_2$.

1.3.2 Parallel Equivalence

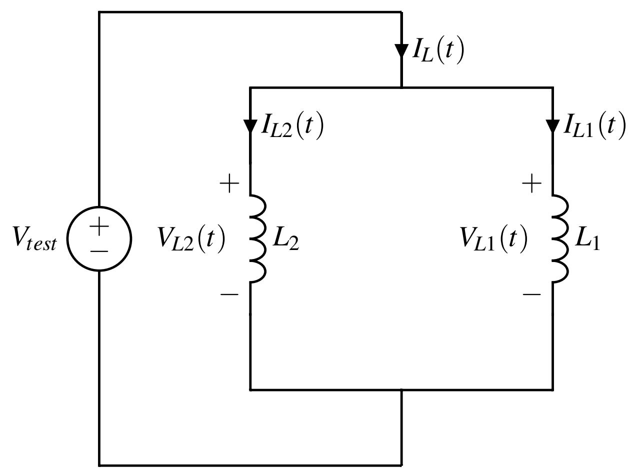


Figure 4: Parallel Inductor Circuit

We apply at V_{test} across the parallel combination. We have

$$V_{L1}(t) = V_{L2}(t) = V_{\text{test}}(t)$$

$$L_1 \frac{dI_{L1}}{dt} = L_2 \frac{dI_{L2}}{dt} = L_{\text{eq}} \frac{dI_L}{dt}$$

and from KCL, we have

$$I_L(t) = I_{L1}(t) + I_{L2}(t)$$

Differentiating with respect to time, and substituting from the above equality,

$$\frac{dI_L}{dt} = \frac{dI_{L1}}{dt} + \frac{dI_{L2}}{dt} \tag{5}$$

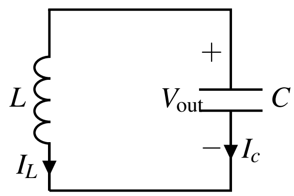
$$\frac{dI_L}{dt} = \frac{L_{\text{eq}}}{L_1}\frac{dI_L}{dt} + \frac{L_{\text{eq}}}{L_2}\frac{dI_L}{dt} \tag{6}$$

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} \tag{7}$$

2 LC Tank

In our two capacitor circuit example, we found that our eigenvalues were real. But, we could also encounter a system whose eigenvalues are complex. In this section, we will explore a circuit, commonly known as an LC tank, whose matrix will have purely imaginary eigenvalues.

In the following circuit, we have an inductor $L = 10\text{nH}$ and capacitor $C = 10\text{pF}$ in parallel. Let $I_L(0) = 50\text{mA}$ and $V_{\text{out}}(0) = 0\text{V}$:



Since the inductor and capacitor are in parallel:

$$V_L = V_C = V_{\text{out}}$$

KCL gives:

$$\begin{aligned} I_L = -I_c &= -C\frac{dV_{\text{out}}}{dt} \\ \frac{dV_{\text{out}}}{dt} &= -\frac{1}{C}I_L \\ V_L = V_{\text{out}} &= L\frac{dI_L}{dt} \\ \frac{dI_L}{dt} &= \frac{1}{L}V_{\text{out}} \end{aligned}$$

Putting it into matrix form, as before:

$$\begin{bmatrix} \frac{d}{dt}V_{\text{out}} \\ \frac{d}{dt}I_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix} \tag{8}$$

Finding the eigenvalues:

$$\det\left(\begin{bmatrix} -\lambda & -\frac{1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{LC} = 0 \tag{9}$$

$$\implies \lambda_{1,2} = 0 \pm j\frac{1}{\sqrt{LC}} \tag{10}$$

Next, we can find the eigenvectors of the above matrix as $v_1 = \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$. We can use these vectors to transform our coordinates to one where the matrix becomes diagonal. More concretely,

$$\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \widetilde{V}_{\text{out}} \\ \widetilde{I}_L \end{bmatrix}$$

As discussed before, once in this new coordinates, our system becomes uncoupled, and we can solve for V_{out} and I_L as follows:

$$\begin{bmatrix} \frac{d}{dt}\widetilde{V}_{\text{out}} \\ \frac{d}{dt}\tilde{I}_L \end{bmatrix} = \begin{bmatrix} j\frac{1}{\sqrt{LC}} & 0 \\ 0 & -j\frac{1}{\sqrt{LC}} \end{bmatrix} \begin{bmatrix} \widetilde{V}_{\text{out}} \\ \tilde{I}_L \end{bmatrix}$$

$$\implies \frac{d}{dt}\widetilde{V}_{\text{out}} = j\frac{1}{\sqrt{LC}}\widetilde{V}_{\text{out}}$$

$$\frac{d}{dt}\tilde{I}_L = -j\frac{1}{\sqrt{LC}}\tilde{I}_L$$

$$\therefore \widetilde{V}_{\text{out}} = \widetilde{k_1}e^{\frac{j}{\sqrt{LC}}t}$$

$$\tilde{I}_L = \widetilde{k_2}e^{-\frac{j}{\sqrt{LC}}t}$$

Next, we need to find initial conditions in this new coordinate system. Substituting the given values,

$$\begin{bmatrix} \widetilde{V}_{\text{out}}(0) \\ \tilde{I}_L(0) \end{bmatrix} = \begin{bmatrix} j\sqrt{\frac{L}{C}} & -j\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} V_{\text{out}}(0) \\ I_L(0) \end{bmatrix}$$

$$= \frac{1}{j20\sqrt{10}} \begin{bmatrix} 1 & j10\sqrt{10} \\ -1 & j10\sqrt{10} \end{bmatrix} \begin{bmatrix} 0 \\ 0.05 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5\times 10^{-2} \\ 2.5\times 10^{-2} \end{bmatrix}$$

Hence, $\widetilde{k_1} = 2.5\times 10^{-2}$ and $\widetilde{k_2} = 2.5\times 10^{-2}$. Next, we can tranform back to our original coordinate system:

$$\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix} = \begin{bmatrix} j10\sqrt{10} & -j10\sqrt{10} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2.5\times 10^{-2}e^{j\sqrt{10}\times 10^9 t} \\ 2.5\times 10^{-2}e^{-j\sqrt{10}\times 10^9 t} \end{bmatrix}$$

$$= \begin{bmatrix} j0.25\sqrt{10}e^{j\sqrt{10}\times 10^9 t} - j0.25\sqrt{10}e^{-j\sqrt{10}\times 10^9 t} \\ 2.5\times 10^{-2}e^{j\sqrt{10}\times 10^9 t} + 2.5\times 10^{-2}e^{-j\sqrt{10}\times 10^9 t} \end{bmatrix}$$

Concept Check: Write the above sum of exponentials as sine and cosine. *Hint: Use the Euler form of sine and cosine we encountered in the complex number note.*

Based on the intuition we have gained above, let's guess a solution with pure sines and cosines, as follows:

$$V_{\text{out}}(t) = c_1\cos\left(\frac{1}{\sqrt{LC}}t\right) + c_2\sin\left(\frac{1}{\sqrt{LC}}t\right) \tag{11}$$

Next, plugging in initial conditions to solve for the constants:

$$V_{\text{out}}(0) = 0 = c_1$$

$$I_c(0) = -I_L(0) = -50\times 10^{-3}$$

$$\frac{dV_{\text{out}}(0)}{dt} = \frac{1}{C}I_c(0) = \frac{-50\times 10^{-3}}{10^{-11}} = \frac{c_2}{\sqrt{10^{-8}\times 10^{-11}}}$$

$$c_1 = 0$$

$$\implies c_2 = -\frac{5}{\sqrt{10}} = -0.5\sqrt{10}$$

$$\implies V_{\text{out}}(t) = -0.5\sqrt{10}\sin\left(\sqrt{10}\times 10^9 t\right)$$

Notice that the amplitude of V_{out} is constant.[1]

Concept Check: Follow the same steps above to find the current, $I_L(t)$. *Hint: The current will also be of the form in eq.* (11)*, but with different constants.*

Solution:

$$I_L(t) = 50 \times 10^{-3} \cos\left(\sqrt{10} \times 10^9 t\right)$$

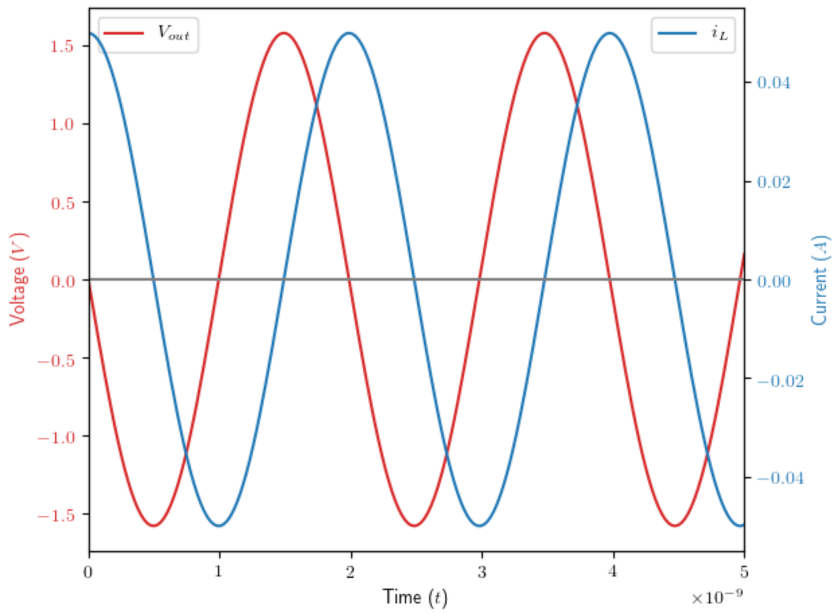


Figure 5: Voltage and Current response of LC Tank

Figure 5 plots the above solutions for the capacitor voltage and inductor current. This system is also called an oscillator because the circuit produces a repetitive voltage waveform under the right initial conditions.

From the above plots, we can see that the current and voltage are 90° out of phase, i.e. when the current is at its maximum or minimum, the voltage is at 0V, and vice versa. What does this mean for the energy stored in these components? We know that, energy in the capacitor, $E_C = \frac{1}{2}CV^2 = 1.25 \times 10^{-11} \sin^2\left(\sqrt{10} \times 10^9 t\right)$ and energy in the inductor, $E_L = 1.25 \times 10^{-11} \cos^2\left(\sqrt{10} \times 10^9 t\right)$. Figure 6 plots the these energies. As you can see, the total energy seems to be sloshing back and forth between the inductor and capacitor.

[1] And, in case the algebra is confusing, the $\frac{c_2}{\sqrt{10^{-8} \times 10^{-11}}}$ part comes from evaluating the derivative of the output voltage at time $t = 0$. That is, $\frac{d}{dt}V_{\text{out}}(t) = c_2 \cos\left(\frac{1}{\sqrt{LC}}t\right)$, and we know the value for this at $t = 0$. Plugging in L, C, gives this equation.

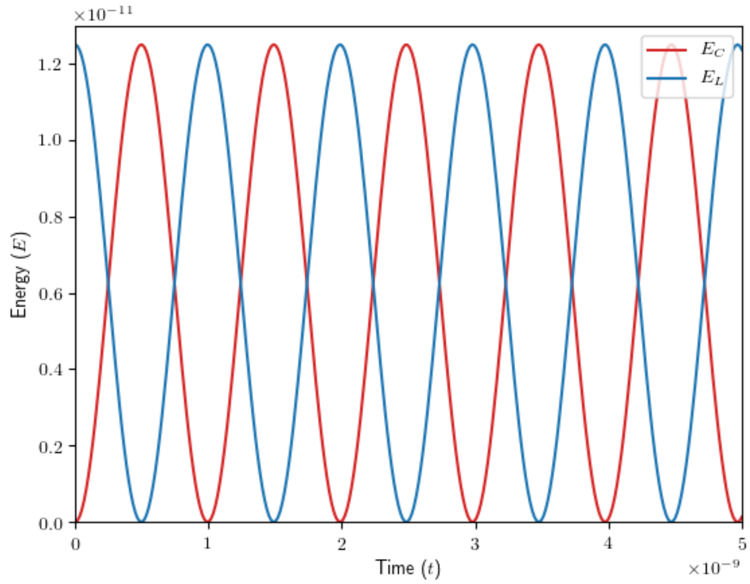
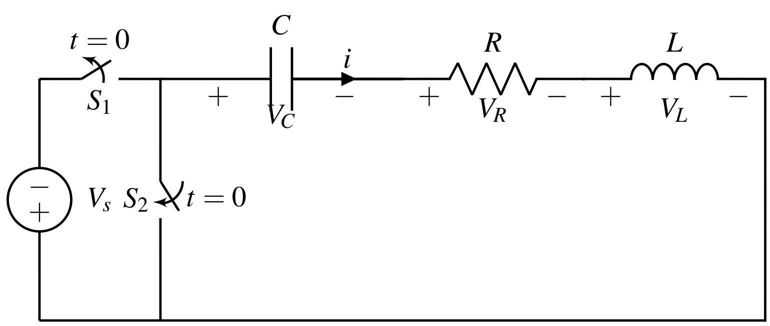


Figure 6: Energy stored in Inductor and Capacitor. Notice the sum is constant.

3 RLC Circuits and Higher Order Differential Equations

The LC tank we studied in the previous section was a very ideal case where we assumed there was no resistor in the system. But this is rarely the case, and we will need to understand how adding this third component will modify our differential equations.

To motivate our discussions, consider the following circuit, with component values $V_s = 4\text{V}$, $C = 2\text{fF}$, $R = 60\text{k}\Omega$, and $L = 1\mu\text{H}$. Before $t = 0$, switch S_1 is on while S_2 is off. At $t = 0$, both switches flip state (S_1 turns off and S_2 turns on):



This is something that you will work out for yourself in the homework. The key is simply to follow the steps marking anything with a derivative on it as a state variable, writing out the differential equations, and solving the system.

Contributors:

- Neelesh Ramachandran.

- Kristofer Pister.
- Utkarsh Singhal.
- Aditya Arun.
- Kyle Tanghe.
- Anant Sahai.
- Kareem Ahmad.
- Nikhil Shinde.