EECS 16B Designing Information Devices and Systems II
Spring 2016 Anant Sahai and Michel Maharbiz — Midterm 1

Exam location: 1 Pimentel (Odd SID + 61C)

PRINT your student ID: ____

PRINT AND SIGN your name: ________________, ________________ ____________________
(last) (first) (signature)

PRINT your Unix account login: ee16b-_____________

PRINT your discussion section and GSI (the one you attend): ____

Row Number (front row is 1): __________ Seat Number (left most is 1): __________

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What other courses are you taking this term? (1 pt)**

2. **Think back to remember a moment where you triumphed. Describe it. How did you feel? (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 1: Straightforward questions (36 points)

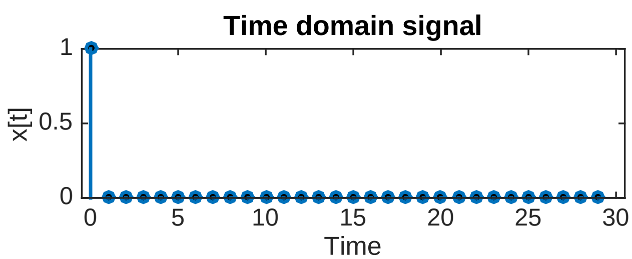
Unless told otherwise, you must show work to get credit. There will be very little partial credit given in this section. **You get one drop: do 4 out of the following 5 questions. (We will grade all and keep the best 4 scores.)** *Each problem is worth 9 points. No additional bonus for getting them all right.*

3. DFT Matching

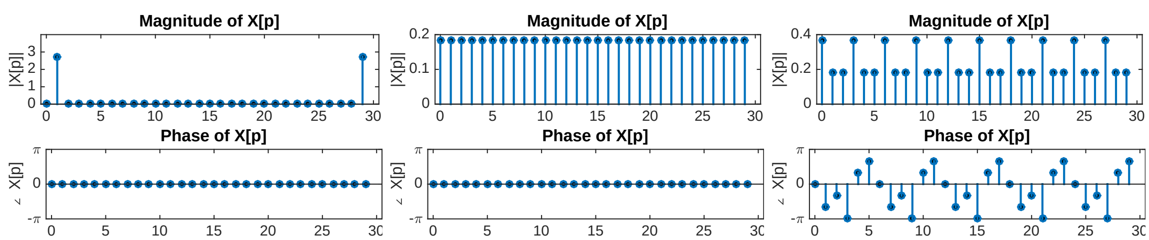
You have to match the n-long time domain signal $\vec{x}$ with its frequency domain $\vec{X}$ coordinates. The DFT coefficients (coordinates in the DFT basis) are represented with their magnitudes $|X[p]|$ and phases $\angle X[p]$, i.e., $X[p] = |X[p]|e^{i\angle X[p]}$ as p goes from 0 (constant) up through $n-1$. Throughout this problem $n = 30$. (For your information, $\frac{1}{\sqrt{30}} \approx 0.18$ and $\sqrt{30} \approx 5.5$.)

Circle your answer. There is no need to give any justification.

(a) Given the time domain signal below,



which one is the corresponding DFT coefficients?



(A) (B) (C)

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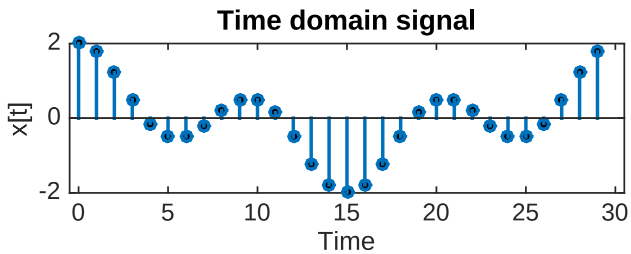
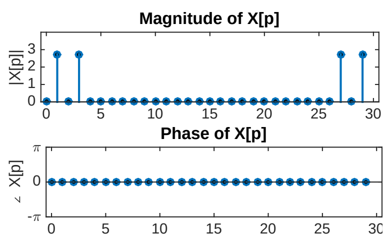
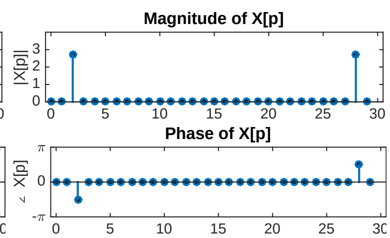
(b) Given the time domain signal below,



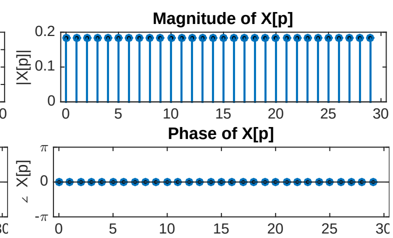
which one is the corresponding DFT coefficients?



(A)

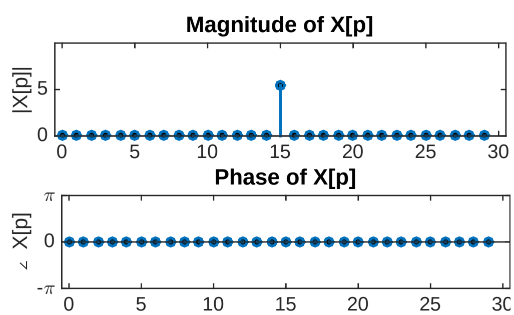


(B)

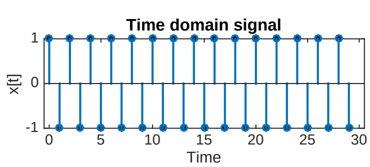


(C)

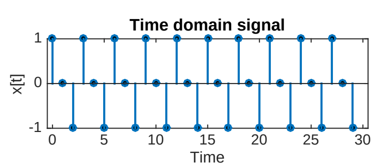
(c) Given the **DFT domain representation** below,



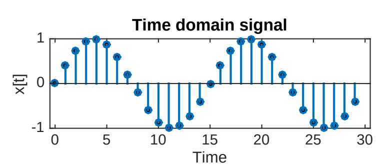
which one is the corresponding **time domain** signal?



(A)



(B)



(C)

4. U square

Consider the 4×4 matrix U that consists of the DFT basis for four-dimensional space. ($n = 4$)

$$U = \begin{bmatrix} \vec{u}_0 & \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$$

(a) **Compute both $\vec{u}_1^{\,T}\vec{u}_2$ and $\vec{u}_1^{\,T}\vec{u}_3$.**

(b) **Compute $U^2 = UU$.**
(HINT: Think about the previous part.)

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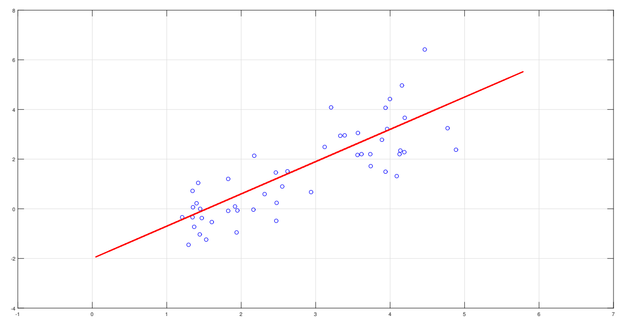
5. PCA Recipe

Let $\vec{x}_i \in \mathbb{R}^d,\ i = 0, \ldots, n-1$ be n data points (column vectors) each having dimension d. Suppose we want to analyze the low dimensional (say k dimensional where $k < d$) structure of this data set with PCA.

The goal is to have just k-coordinates for each data point that represent where they are relative to each other on the best k-dimensional structure that fits all the data points. The following figure provides an illustration with $k = 1$ and $d = 2$.



In the figure, the line is the essential 1 dimensional structure and the dots are the n data points. For each data point, we would want to know the one coordinate describing how far along the line that data point's closest representative was.

Your task is to arrange *in order* an appropriate subset of the steps below to make that happen. Note that not all steps listed will be used.

Just write out a string (e.g. LEBIGMA) as your answer.

A. Assemble the vectors $\{\vec{y}_i\}$ into the columns of the matrix Y.

B. Run SVD on the matrix Y to get $Y = U\Sigma V^T$.

C. Copy the data points into vectors $\vec{y}_i = \vec{x}_i$.

D. Subtract the average from each dimension of the data: $y_i[j] = x_i[j] - \frac{1}{n}\sum_{\ell=0}^{n-1} x_\ell[j]$.

E. Subtract the average from each data point: $y_i[j] = x_i[j] - \frac{1}{d}\sum_{t=0}^{d-1} x_i[t]$.

F. Multiply the matrix Y by V'^T on the right. i.e. Compute YV'^T to get the desired coordinates.

G. Multiply the matrix Y by V'^T on the left. i.e. Compute $V'^T Y$ to get the desired coordinates.

H. Multiply the matrix Y by U'^T on the right. i.e. Compute YU'^T to get the desired coordinates.

I. Multiply the matrix Y by U'^T on the left. i.e. Compute $U'^T Y$ to get the desired coordinates.

J. Select the first k columns from the matrix V to get V'

K. Select the last k columns from the matrix V to get V'

L. Select the first k columns from the matrix U to get U'

M. Select the last k columns from the matrix U to get U'

6. DFT Meets k-means

There are n discrete-time signals, $\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_{n-1}$, where the length of each signal is d. We want to use the k-means algorithm to classify them into k clusters, where $k < n$.

Suppose that we run k-means on the time-domain signals $\vec{x}_0, \ldots, \vec{x}_{n-1}$ to find k clusters using the first k signals to initialize the cluster centers.

Alternatively, suppose we run k-means on the same vectors except in frequency domain (i.e. we take the DFT of each signal and then run k-means on that collection of DFT-ed signals) using the frequency-domain representation of the first k signals to initialize the cluster centers.

Is the resulting classification of the n signals by running k-means in the frequency domain going to be the same or different than the classification that results from running k-means in the time domain? Briefly justify your answer.

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7. SVD

Let $A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}^{T}$.

Find $\vec{x}$ such that

$$A\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and $\|\vec{x}\|$ is as small as possible.

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 2: Free-form Problems (72 + 10 points)

8. ROC K-Means (15 pts)

This question is about running k-means to group people into two rock bands based on musical taste. We start with two people as "band centers" to initialize the algorithm. People regroup into bands based on which band has an average musical taste closer to them.

This process of determining the musical taste for each band and re-choosing the groups is called re-grouping. As in k-means, re-grouping will continue until nobody wants to leave their band and join the other one. At that point, we will have bands that are ready to actually play gigs.

We decide to monitor the grouping process as k-means evolves. To do this, we define the state to be the way people are partitioned into two groups. It turns out that with this grouping process, it is impossible to have an empty band. Each band will always have at least one person in it.

In this problem, we consider four people, Ambika, Bao, Carol and Diego, the number of possible states is 7. We label them in the table below.

State	Combination
S_0	$\{[A],[B,C,D]\}$
S_1	$\{[B],[A,C,D]\}$
S_2	$\{[C],[A,B,D]\}$
S_3	$\{[D],[A,B,C]\}$
S_4	$\{[A,B],[C,D]\}$
S_5	$\{[A,C],[B,D]\}$
S_6	$\{[A,D],[B,C]\}$

Notice that $\{[A],[B,C,D]\}$ and $\{[D,C,B],[A]\}$ are the same state because a band is defined by its members.

For simplicity, suppose that musical taste is unidimensional — all that matters is how much you like cowbells. The cowbell scores are listed below:

Name	Ambika	Bao	Carol	Diego
Score	90	85	70	40

(a) (5 pts) **Based on the cowbell scores above, can some initial choice of two people lead to k-means starting in the state S_0? How about the state S_1? How about the state S_3?** Just tell us which of these are possible starting states and which are impossible.

(b) (5 pts) To track the evolution of the band-formation process, we decide to use a *state transition matrix* T. In this matrix, if $T_{ij} = 1$, it means the group state will change from S_j to S_i if one step of re-grouping happens. In other words, if we compute $T\vec{e}_j$, where $\vec{e}_j$ is the standard j-th basis vector, we will get $\vec{e}_i$. Consider a hypothetical three-state example:

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

In this example, S_0 will become S_1, S_1 will become S_2, and S_2 will stay forever.

Please fill in the missing entries in the transition matrix below for the Ambika, Bao, Carol and Diego example according to the cowbell scores given previously and the seven states as ordered earlier.

$$T = \begin{bmatrix} 0 & 0 & 0 & \square & 0 & 0 & 0 \\ 0 & 0 & 0 & \square & 0 & 0 & 0 \\ 0 & 0 & 0 & \square & 0 & 0 & 0 \\ 0 & 0 & 0 & \square & 0 & 0 & 0 \\ 1 & 1 & 1 & \square & 1 & 1 & 1 \\ 0 & 0 & 0 & \square & 0 & 0 & 0 \\ 0 & 0 & 0 & \square & 0 & 0 & 0 \end{bmatrix}$$

(c) (5 pts) **According to the matrix you just created, what are all the possible final states for the bands?**

9. Echos revisited (27 pts)

Recall that in the real-world, the radio waves emitted by a wireless transmitter don't only go directly to the wireless receiver that wants to listen to them. Instead, they propagate physically, much the same way that sound propagates. If there is a clear line-of-sight path between the transmitter and the receiver, then certainly the radio waves do reach the destination directly. But they also find other paths, bouncing off of obstacles like walls, the ground, the ceiling, buildings, hills, etc. Because the speed of light is finite, all of these echoes of the transmitted signal reach the destination with different delays.

In this problem, the wireless channel has a direct path with gain 1 and single echo t_d later with gain 0.8.

As we have in the 16 series so far, we will work in finite time. The total length of the signal is denoted by n since as in the locationing labs, the transmissions are assumed to be periodic with period n. So the time-delay $t_d \in \{0,1,2,\ldots,n-1\}$ and the wireless channel's impulse response $\vec{h}$ has $h[0] = 1$ and $h[t_d] = 0.8$ with all other $h[t] = 0$.

Let the signal $\vec{x}$ be some signal sent by the transmitter, and the signal $\vec{y}$ be the one observed at the receiver. The two signals are connected based on the equation $\vec{y} = C_{\vec{h}}\vec{x}$, where $C_{\vec{h}}$ is the circulant matrix that has $\vec{h}$ as its first column.

Use $n = 5$ for this entire problem.

(a) (6 pts) Let the frequency-domain representations (coordinates in DFT basis) of $\vec{x}$, $\vec{y}$, $\vec{h}$ be $\vec{X}$, $\vec{Y}$, $\vec{H}$, respectively.
Express $\vec{Y}$ in terms of $\vec{H}$ and $\vec{X}$?

(b) (5 pts) Let the DFT coefficients of $\vec{h}$ be $\vec{H}$. **Write down the p-th element of $\vec{H}$ in terms of t_d, p, and $n = 5$.**

(c) (5 pts) To figure out what the impulse response ($\vec{h}$) is, we decide to send a cosine signal $\vec{x}$ at the transmitter and observe the signal $\vec{y}$ at the receiver. Let the t-th element of $\vec{x}$, $x[t]$, be $\cos\left(\frac{2\pi}{5}t\right)$. That is,

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{5}(0)\right) & \cos\left(\frac{2\pi}{5}(1)\right) & \cos\left(\frac{2\pi}{5}(2)\right) & \cos\left(\frac{2\pi}{5}(3)\right) & \cos\left(\frac{2\pi}{5}(4)\right) \end{bmatrix}^T. \tag{1}$$

What is the frequency domain representation (coordinates in the DFT basis) of $\vec{x}$?

(d) (5 pts) After sending the above signal $\vec{x}$, we observe a signal $\vec{y}$ at the receiver. The DFT coefficients of $\vec{y}$ are given by

$$\vec{Y} = \frac{\sqrt{5}}{2} \begin{bmatrix} 0 \\ 1+0.8e^{-i\frac{4\pi}{5}} \\ 0 \\ 0 \\ 1+0.8e^{i\frac{4\pi}{5}} \end{bmatrix} \tag{2}$$

What is the echo delay, t_d, of $\vec{h}$?

(e) (6 pts) **Please write the t-th element of the time-domain $\vec{y}$ from above in the form of $y[t] = \alpha \cos\left(\frac{2\pi}{5}t + \theta\right)$.**

What is θ? What is α? (You are permitted to use the Angle function to convert a complex number into polar coordinates and you don't have to simplify your answers.)

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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10. Reverse-engineering (30 + 10pts)

One day, a small black box catches your eye. You find a label on the box that reads "Acme SportsRank Proprietary Repeated Multiplier" and learn online that the black box contains some **square** $n \times n$ **symmetric matrix** A with strictly positive eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_{n-1} > 0$. You can't look at the entries in the matrix, but you can compute $A^k\vec{x}$ for any $\vec{x}$ that you choose and any $k > 20$. You know nothing else about A due to its proprietary nature. In this problem, you will find the eigenvectors of A so that you can reverse-engineer the matrix.

(a) (15 pts) Our first task is to find the eigenvector $\vec{u}_0$ corresponding with the largest eigenvalue λ_0. To find this eigenvector we start with some randomly generated vector $\vec{x}$ and use the box to repeatedly multiply $\vec{x}$ by the matrix A. You can assume that anytime you randomly chose a vector $\vec{x}$, it is guaranteed to be able to be written as $\vec{x} = \sum_{p=0}^{n-1} \beta_p \vec{u}_p$ where $\vec{u}_p$ is the normalized p-th eigenvector of A with all the $|\beta_p| > 0$ for all p.

Show that

$$\lim_{k\to\infty} \frac{A^k\vec{x}}{||A^k\vec{x}||} = \vec{u}_0.$$

(HINT: Use the fact that $\left(\frac{x}{y}\right)^k \to 0$ *when* $x < y$ *and* k *is large.)*

(b) (15 pts) The argument above tells us that we can essentially compute a normalized eigenvector as $\vec{u}_0 = \frac{A^k\vec{x}}{||A^k\vec{x}||}$ by choosing a sufficiently large k. Assume that this works perfectly (neglect the effect of k being finite). Now that we have found $\vec{u}_0$, we want to find the eigenvector corresponding with the second largest eigenvalue λ_1. **Use what you know about the eigenvectors for a symmetric matrix as well as the method from the previous part to describe how we could find $\vec{u}_1$ using only black-box access to multiplication by A.** (You cannot change the matrix A — all you can do is pick vectors to multiply by it k times. You are allowed to get a new random vector if you'd like, as well as do any operations on it that you want involving $\vec{u}_0$.) Argue briefly why your method should work.

(c) (Bonus 10pts) Extend the previous part to create a way to find all of the eigenvectors of A. Show how to do the key inductive step. **Assuming we have found the first i eigenvectors, describe a method we can use to find the next eigenvector $\vec{u}_{i+1}$.**

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[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]